

THE BLOCH-OKOUNKOV CORRELATION FUNCTIONS OF CLASSICAL TYPE

DAVID G. TAYLOR AND WEIQIANG WANG

ABSTRACT. Bloch and Okounkov introduced an n -point correlation function on the infinite wedge space and found an elegant closed formula in terms of theta functions. This function has connections to Gromov-Witten theory, Hilbert schemes, symmetric groups, etc, and it can also be interpreted as correlation functions on integrable $\widehat{\mathfrak{gl}}_\infty$ -modules of level one. Such $\widehat{\mathfrak{gl}}_\infty$ -correlation functions at higher levels were then calculated by Cheng and Wang.

In this paper, generalizing the type A results, we formulate and determine the n -point correlation functions in the sense of Bloch-Okounkov on integrable modules over classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ of type B, C, D at arbitrary levels. As byproducts, we obtain new q -dimension formulas for integrable modules of type B, C, D and some fermionic type q -identities.

CONTENTS

1. Introduction	2
1.1. The earlier works	2
1.2. The goal	3
1.3. Our approach	3
1.4. Open questions	4
1.5. Organization and Acknowledgment	5
2. The preliminaries	5
2.1. Classical Lie algebras of infinite dimension	6
2.2. Classical Lie groups	8
2.3. Additional notations	11
3. Correlation functions on d_∞ -modules of level l	11
3.1. The Fock space \mathcal{F}^l	11
3.2. The $(O(2l), d_\infty)$ -Howe duality	12
3.3. The main results of [BO, CW1]	13
3.4. The 1-point d_∞ -functions of level l	15
3.5. The n -point d_∞ -functions of level l	18
3.6. A refined 1-point function of level 1	19
3.7. The q -dimension of a d_∞ -module of level l	22

4.	Correlation functions on d_∞ -modules of level $l + \frac{1}{2}$	23
4.1.	The Fock space $\mathcal{F}^{l+\frac{1}{2}}$	23
4.2.	The $(O(2l+1), d_\infty)$ -Howe duality	24
4.3.	The n -point d_∞ -function of level $\frac{1}{2}$	25
4.4.	The n -point d_∞ -functions of level $l + \frac{1}{2}$	27
4.5.	The q -dimension of a d_∞ -module of level $l + \frac{1}{2}$	28
5.	Correlation functions on c_∞ -modules of level l	29
5.1.	The $(Sp(2l), c_\infty)$ -Howe duality	29
5.2.	The n -point c_∞ -functions of level l	30
5.3.	The q -dimension of a c_∞ -module	31
6.	Correlation functions on b_∞ -modules of level l	32
6.1.	The $(Pin(2l), b_\infty)$ -Howe duality	32
6.2.	The operator $B(t)$	32
6.3.	The n -point b_∞ -functions of level l	33
6.4.	The q -dimension of a b_∞ -module	35
7.	Correlation functions on b_∞ -modules of level $l + \frac{1}{2}$	35
7.1.	The $(Spin(2l+1), b_\infty)$ -Howe duality	35
7.2.	The n -point b_∞ -function of level $\frac{1}{2}$	36
7.3.	The n -point b_∞ -functions of level $l + \frac{1}{2}$	38
7.4.	The q -dimension of a b_∞ -module of level $l + \frac{1}{2}$	39
	References	39

1. INTRODUCTION

1.1. The earlier works. Bloch and Okounkov [BO] (also see [Ok] for some simplification) introduced an n -point correlation function on the infinite wedge space and found an elegant closed formula in terms of theta functions. Their work was in part motivated by certain modular invariance property of trace functions of vertex operators and representation theory of the $W_{1+\infty}$ algebra (cf. [Zhu, FKRW, Blo]). Subsequently, this function and its variant have been interpreted as a generating function of the Gromov-Witten invariants of an elliptic curve by Okounkov-Pandharipande [OP], and as a generating function of intersection numbers on Hilbert schemes of points by Li, Qin and the second author [LQW]. We also refer the reader to [Lep, Mil] for formal vertex operator generalizations, [W2] for a neutral fermionic Fock space version, and [CW2] for a q, t -deformation of the Bloch-Okounkov n -point function.

From a representation theoretic viewpoint, the Bloch-Okounkov n -point function can be also easily interpreted as correlation functions

on integrable modules over Lie algebra $\widehat{\mathfrak{gl}}_\infty$ of level one (cf. [Ok, Mil, CW1]). Along this line, Cheng and the second author [CW1] formulated and calculated such n -point correlation functions on integrable $\widehat{\mathfrak{gl}}_\infty$ -modules of level l ($l \in \mathbb{N}$).

1.2. The goal. The goal of this paper is to formulate and determine the n -point correlation functions in the sense of Bloch-Okounkov on integrable modules over classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ of type B, C, D at arbitrary level, generalizing the works [BO, CW1] in type A . Note that the integrability of these modules implies that the levels have to be positive (half-)integral. By the original works of Date, Jimbo, Kashiwara and Miwa ([DJKM1, DJKM2]), $\widehat{\mathfrak{gl}}_\infty$ affords classical Lie subalgebras of type B, C, D , and these infinite-dimensional Lie algebras played an important role in connections with soliton equations discovered by the Kyoto school in early 1980's.

The representation theory of $\widehat{\mathfrak{gl}}_\infty$ is intimately related to that of the $W_{1+\infty}$ algebra (cf. [FKRW] and the references therein). It follows that the Bloch-Okounkov correlation functions for $\widehat{\mathfrak{gl}}_\infty$ -modules can be regarded as those for $W_{1+\infty}$ -modules. In the same vein, the representation theory of the classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ is closely related to that of the classical Lie subalgebras of $W_{1+\infty}$ initiated in [KWY]; the Howe dualities [W1], which are to be used extensively in this work, readily carry over if one replaces classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ by classical Lie subalgebras of $W_{1+\infty}$. In this way, the n -point correlation functions studied in this paper can be in turn regarded as those for modules over classical Lie subalgebras of $W_{1+\infty}$.

1.3. Our approach. To achieve our goal, the first (main) step here is to relate the correlation functions at higher levels to the correlation functions at the bottom levels (i.e. of level one and/or level $\frac{1}{2}$). Our main tool is the free field realization [DJKM1, DJKM2] (also cf. Feingold-Frenkel [FF]) and the Howe duality due to the second author [W1] between the classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ and various classical Lie groups (where sometimes disconnected groups and different covering groups are required). We refer to [Ho1, Ho2] for Howe's original setups, where all Lie algebras and groups involved are finite-dimensional.

We note that all integrable modules of these Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ appear in these Howe duality decompositions, and the level of an integrable module matches with the rank of the corresponding Lie group. A detailed knowledge of irreducible modules over various Lie groups (cf. Bröcker-tom Dieck [BtD]) and the determinantal ratio form of the

Weyl character formulas for classical Lie algebras (cf. Fulton-Harris [FH]) are also used in this paper in an essential way.

A similar approach has actually been applied in [CW1] successfully where the type A Howe duality between $\widehat{\mathfrak{gl}}_\infty$ and GL_l due to I. Frenkel [Fr] (also cf. [W1]) was used. Forced by the new technical features in type B, C, D , we establish in this paper the relations between the n -point functions at higher levels and at the bottom levels in a different way, avoiding the usage of the q -dimension formula for integrable modules in [CW1]. As a byproduct, we obtain neat q -dimension formulas for the corresponding integrable modules over the classical Lie subalgebras of type B, C, D , which are simpler than the ones in [KWY] obtained by a specialization of the Weyl-Kac character formulas. We remark that the idea of using Howe duality to obtain irreducible character formulas has also been applicable in different setups (cf. Cheng-Lam [CL]).

Our second step is more straightforward. By using the free field realization we are able to relate the calculation of the n -point function of type B, D of level one to the n -point function of type A of level one which has been computed in [BO]. The type C level one case can be handled by a combination of Howe duality and the connection to the type A level one case.

An additional step is needed to take care of the half-integral levels, which occur in type B and D . Using an identification of a pair of complex fermions and two neutral fermions, we obtain formulas, recursive on n , of computing the n -point functions of classical type of level $\frac{1}{2}$ in terms of those of level one. Explicit formulas in different forms for the 1-point functions of type B and D of level $\frac{1}{2}$ were obtained in [W2] using the method of partition identities. Identifying these different formulas gives rise to two interesting q -identities of fermionic type. It turns out that these identities have been known with a very different proof (cf. e.g. [Kac]).

Combining all these steps together, we have calculated all the n -point correlation functions of classical type. The final formulas involve the Weyl groups of the Lie groups appearing in various Howe dualities and the original Bloch-Okounkov function of type A and level one (which in turn is an expression in terms of theta functions). Remarkably, the solutions in type B and type D look almost identical formally though different Lie algebras and different Howe dualities are involved in different type.

1.4. Open questions. The integrable modules whose correlation functions are computed here are occasionally not irreducible (instead it could be a sum of two irreducibles) over the infinite-dimensional Lie

algebras of type D , but they can always be regarded as irreducible modules over the corresponding orthogonal groups. This is a familiar phenomenon of spinor vs half-spinor representations. Nevertheless, it will be interesting to determine completely the (refined) n -point functions for all *irreducible* integrable modules of type D . In this direction, we have only obtained limited results. By observing an intrinsic connection with the theory of partitions (cf. Andrews [An]), we find an explicit formula for the refined 1-point function of type D of level one.

We can also formulate the n -point correlation functions for modules of *negative* (half-)integral levels of $\widehat{\mathfrak{gl}}_\infty$ and its classical subalgebras, and these modules have appeared in the Howe duality decompositions (cf. [KR] for type A and [W1] in general). It will be interesting to determine these n -point correlation functions. In light of the developments in [CW1] and in this paper, the main difficulty lies in understanding the cases at level -1 , where the connection with the theory of partitions available at level one and at level $\frac{1}{2}$ is now lacking.

A more challenging question is to ask for a geometric interpretation of the correlation functions studied in this paper (and also in [CW1]). For example, can they be interpreted as correlation functions in some supersymmetric gauge theory where the classical Lie groups used in this paper appear as gauge groups?

1.5. Organization and Acknowledgment. The paper is organized as follows. In Section 2, we set up the notations for the classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ and various Lie groups used in later sections. In Sections 3 and 4 respectively, we formulate and calculate the n -point functions and the q -dimension of integrable d_∞ -modules of level l and of level $l + \frac{1}{2}$ respectively. In Section 5, we calculate the n -point functions and the q -dimension of integrable c_∞ -modules of level l . In Sections 6 and 7 respectively, we formulate and calculate the n -point functions and the q -dimension of integrable b_∞ -modules of level l and of level $l + \frac{1}{2}$ respectively.

Various Fock spaces of free fermionic fields and Howe dualities are recalled and used in each of the Sections 3–7. The proofs in Sections 3 and 4 are given in detail, while the proofs in Sections 5–7 are often sketchy when they are parallel to the ones in Sections 3–4.

This research is partially supported by NSF and NSA grants. We thank Shun-Jen Cheng for helpful discussions and comments.

2. THE PRELIMINARIES

The purpose of this section is to set up notations for the infinite-dimensional Lie algebras and classical Lie groups which we will use.

2.1. Classical Lie algebras of infinite dimension. In this subsection we review Lie algebras $\widehat{\mathfrak{gl}} \equiv \widehat{\mathfrak{gl}}_\infty$ and its various Lie subalgebras of B, C, D type (cf. [DJKM1, DJKM2]).

2.1.1. Lie algebra $\widehat{\mathfrak{gl}}$. Denote by \mathfrak{gl} the Lie algebra of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ satisfying $a_{ij} = 0$ for $|i-j|$ sufficiently large. Denote by E_{ij} the infinite matrix with 1 at (i, j) place and 0 elsewhere and let the weight of E_{ij} be $j - i$. This defines a \mathbb{Z} -principal gradation $\mathfrak{gl} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{gl}_j$. Denote by $\widehat{\mathfrak{gl}} \equiv \widehat{\mathfrak{gl}}_\infty = \mathfrak{gl} \oplus \mathbb{C}C$ the central extension given by the following 2-cocycle with values in \mathbb{C} (cf. [DJKM1]):

$$C(A, B) = \text{tr}([J, A]B) \quad (1)$$

where $J = \sum_{j \leq 0} E_{ii}$. The \mathbb{Z} -gradation of Lie algebra \mathfrak{gl} extends to $\widehat{\mathfrak{gl}}$ by letting the weight of C to be 0. This leads to a triangular decomposition

$$\widehat{\mathfrak{gl}} = \widehat{\mathfrak{gl}}_+ \oplus \widehat{\mathfrak{gl}}_0 \oplus \widehat{\mathfrak{gl}}_-$$

where $\widehat{\mathfrak{gl}}_\pm = \bigoplus_{j \in \mathbb{N}} \widehat{\mathfrak{gl}}_{\pm j}$, $\widehat{\mathfrak{gl}}_0 = \mathfrak{gl}_0 \oplus \mathbb{C}C$. Let

$$H_i^a = E_{ii} - E_{i+1, i+1} + \delta_{i,0}C \quad (i \in \mathbb{Z}).$$

Denote by $L(\widehat{\mathfrak{gl}}; \Lambda)$ the highest weight $\widehat{\mathfrak{gl}}$ -module with highest weight $\Lambda \in \widehat{\mathfrak{gl}}_0^*$, where C acts as a scalar which is called the *level*. Let $\Lambda_j^a \in \widehat{\mathfrak{gl}}_0^*$ be the fundamental weights, i.e. $\Lambda_j^a(H_i^a) = \delta_{ij}$. The Dynkin diagram for $\widehat{\mathfrak{gl}}$, with fundamental weights labeled, is the following:

$$\cdots \circ \text{---} \underset{-2}{\circ} \text{---} \underset{-1}{\circ} \text{---} \underset{0}{\circ} \text{---} \underset{1}{\circ} \text{---} \underset{2}{\circ} \cdots$$

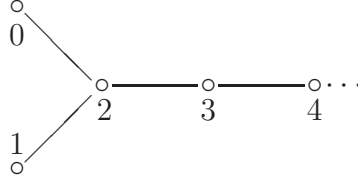
2.1.2. Lie algebra d_∞ . Let

$$\overline{d}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -a_{1-j, 1-i}\}$$

be a Lie subalgebra of \mathfrak{gl} of type D . Denote by $d_\infty = \overline{d}_\infty \oplus \mathbb{C}C$ the central extension given by the 2-cocycle (1). Then d_∞ has a natural triangular decomposition induced from $\widehat{\mathfrak{gl}}$ with Cartan subalgebra $d_{\infty 0} = \widehat{\mathfrak{gl}}_0 \cap d_\infty$. Given $\Lambda \in d_{\infty 0}^*$, we let

$$\begin{aligned} H_i^d &= E_{ii} + E_{-i, -i} - E_{i+1, i+1} - E_{-i+1, -i+1} \quad (i \in \mathbb{N}), \\ H_0^d &= E_{0,0} + E_{-1, -1} - E_{2,2} - E_{1,1} + 2C. \end{aligned}$$

Denote by Λ_i^d the i -th fundamental weight of d_∞ , i.e. $\Lambda_i^d(H_j^d) = \delta_{ij}$. The Dynkin diagram of d_∞ is:



2.1.3. *Lie algebra c_∞ .* Let

$$\bar{c}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -(-1)^{i+j} a_{1-j, 1-i}\}$$

be a Lie subalgebra of \mathfrak{gl} of type C . Denote by c_∞ the central extension of \bar{c}_∞ given by the 2-cocycle (1). Then c_∞ inherits from $\widehat{\mathfrak{gl}}$ a natural triangular decomposition with Cartan subalgebra $c_{\infty 0}$. Given $\Lambda \in c_{\infty 0}^*$, we let

$$\begin{aligned} H_i^c &= E_{ii} + E_{-i, -i} - E_{i+1, i+1} - E_{1-i, 1-i} \quad (i \in \mathbb{N}), \\ H_0^c &= E_{0,0} - E_{1,1} + C. \end{aligned}$$

Denote by Λ_i^c the i -th fundamental weight of c_∞ , i.e. $\Lambda_i^c(H_j^c) = \delta_{ij}$. The Dynkin diagram of c_∞ is:



2.1.4. *Lie algebra b_∞ .* Let

$$\bar{b}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -a_{-j, -i}\}$$

be a Lie subalgebra of \mathfrak{gl} of type B . Denote by b_∞ the central extension of \bar{b}_∞ given by the 2-cocycle (1). The Lie algebra b_∞ inherits from $\widehat{\mathfrak{gl}}$ a natural triangular decomposition with Cartan subalgebra $b_{\infty 0}$. Given $\Lambda \in b_{\infty 0}^*$, we let

$$\begin{aligned} H_i^b &= E_{ii} + E_{-i-1, -i-1} - E_{i+1, i+1} - E_{-i, -i} \quad (i \in \mathbb{N}), \\ H_0^b &= 2(E_{-1, -1} - E_{1,1}) + 2C. \end{aligned}$$

Denote by Λ_i^b the i -th fundamental weight of b_∞ , i.e. $\Lambda_i^b(H_j^b) = \delta_{ij}$. The Dynkin diagram of b_∞ is:



2.2. Classical Lie groups. We present here a parametrization of irreducible modules of various classical Lie groups. See [BtD] (also [W1]) for more detail.

2.2.1. $O(2l)$. We define $O(2l) = \{g \in GL(2l); {}^t g J g = J\}$ with

$$J = \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix}$$

Lie group $GL(l)$ can be identified as a subgroup of $O(2l)$ consisting of matrices of the form $\text{diag}(g, {}^t g^{-1})$, where ${}^t g$ denotes the transpose. Lie algebra $\mathfrak{so}(2l)$ of $SO(2l)$ consists of matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ \gamma & -{}^t \alpha \end{bmatrix} \quad (2)$$

where α, β, γ are $l \times l$ matrices and β, γ are skew-symmetric. Lie algebra $\mathfrak{gl}(l)$ is identified with the subalgebra of $\mathfrak{so}(2l)$ consisting of matrices of the form (2) with $\beta = \gamma = 0$. Let $\mathfrak{b}(\mathfrak{so}(2l))$ be the Borel subalgebra of $\mathfrak{so}(2l)$ which consists of matrices (2) with $\gamma = 0$ and α upper triangular matrices, and $\mathfrak{h}(\mathfrak{so}(2l))$ be the Cartan subalgebra of diagonal matrices $\text{diag}(t_1, \dots, t_l, -t_1, \dots, -t_l)$, $t_i \in \mathbb{C}$. Then $\mathfrak{gl}(l)$ and $\mathfrak{so}(2l)$ share the same Cartan subalgebra.

An irreducible $GL(l)$ -module is parameterized by its highest weight which runs over the set

$$\Sigma(A) \equiv \{(m_1, m_2, \dots, m_l) \mid m_1 \geq m_2 \geq \dots \geq m_l, m_i \in \mathbb{Z}\}.$$

An irreducible $SO(2l)$ -module is parameterized by its highest weight in $\{(m_1, m_2, \dots, m_l) \mid m_1 \geq m_2 \geq \dots \geq m_{l-1} \geq |m_l|, m_i \in \mathbb{Z}\}$. For notational simplicity, we may identify a highest weight module with its highest weight below.

$O(2l)$ is a semi-direct product of $SO(2l)$ by \mathbb{Z}_2 . Denote by $\tau \in O(2l) \setminus SO(2l)$ the $2l \times 2l$ matrix

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad (3)$$

with $A = \text{diag}(1, \dots, 1, 0)$, $B = \text{diag}(0, \dots, 0, 1)$. Then τ normalizes the Borel \mathfrak{b} . If λ is an $SO(2l)$ -module of highest weight (m_1, m_2, \dots, m_l) , then $\tau \cdot \lambda$ has highest weight $\bar{\lambda} := (m_1, m_2, \dots, -m_l)$. The induced module of (m_1, m_2, \dots, m_l) ($m_l \neq 0$) to $O(2l)$ is irreducible and its restriction to $SO(2l)$ is a sum of (m_1, m_2, \dots, m_l) and $(m_1, m_2, \dots, -m_l)$. We denote this $O(2l)$ -module λ by (m_1, m_2, \dots, m_l) , where $m_l > 0$. If $m_l = 0$, the module $\lambda = (m_1, m_2, \dots, m_{l-1}, 0)$ extends to two different

$O(2l)$ -modules, denoted by λ and $\lambda \otimes \det$, where \det is the 1-dimensional non-trivial $O(2l)$ -module. We denote

$$\begin{aligned} \Sigma(D) = & \{(m_1, m_2, \dots, m_l) \mid m_1 \geq m_2 \geq \dots \geq m_l > 0, m_i \in \mathbb{Z}\} \\ & \cup \{(m_1, m_2, \dots, m_{l-1}, 0) \otimes \det, (m_1, m_2, \dots, m_{l-1}, 0) \mid \\ & m_1 \geq m_2 \geq \dots \geq m_{l-1} \geq 0, m_i \in \mathbb{Z}\}. \end{aligned}$$

2.2.2. $O(2l+1)$. Let $O(2l+1) = \{g \in GL(2l+1) \mid {}^t g J g = J\}$, where

$$J = \begin{bmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Lie algebra $\mathfrak{so}(2l+1)$ is the Lie subalgebra of $\mathfrak{gl}(2l+1)$ consisting of $(2l+1) \times (2l+1)$ matrices of the form

$$\begin{bmatrix} \alpha & \beta & \delta \\ \gamma & -{}^t\alpha & h \\ -{}^th & -{}^t\delta & 0 \end{bmatrix} \quad (4)$$

where α, β, γ are $l \times l$ matrices and β, γ skew-symmetric. The Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l+1))$ consists of matrices (4) by putting γ, h, δ to be 0 and α to be upper triangular. The Cartan subalgebra $\mathfrak{h}(\mathfrak{so}(2l+1))$ consists of diagonal matrices of the form $\text{diag}(t_1, \dots, t_l; -t_1 \dots - t_l; 0)$, $t_i \in \mathbb{C}$. An irreducible module of $SO(2l+1)$ is parameterized by its highest weight $(m_1, \dots, m_l) \in \mathcal{P}^l$, where \mathcal{P}^l denotes the set of partitions with at most l non-zero parts.

It is well known that $O(2l+1)$ is isomorphic to the direct product $SO(2l+1) \times \mathbb{Z}_2$ by sending the minus identity matrix to $-1 \in \mathbb{Z}_2 = \{\pm 1\}$. Denote by \det the non-trivial one-dimensional representation of $O(2l+1)$. An representation λ of $SO(2l+1)$ extends to two different representations λ and $\lambda \otimes \det$ of $O(2l+1)$. Then we can parameterize irreducible representations of $O(2l+1)$ by (m_1, \dots, m_l) and $(m_1, \dots, m_l) \otimes \det$. We shall denote

$$\Sigma(B) = \mathcal{P}^l \cup \{\lambda \otimes \det \mid \lambda \in \mathcal{P}^l\}.$$

2.2.3. *Spin*(n) and *Pin*(n). The Pin group $Pin(n)$ is the double covering group of $O(n)$, namely we have

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Pin(n) \longrightarrow O(n) \longrightarrow 1.$$

We then define the spin group $Spin(n)$ to be the inverse image of $SO(n)$ under the projection from $Pin(n)$ to $O(n)$.

Case $n = 2l$. Let $\mathbf{1}_l = (1, 1, \dots, 1)$, $\bar{\mathbf{1}}_l = (1, 1, \dots, 1, -1) \in \mathbb{Z}^l$. An irreducible representation of $Spin(2l)$ which does not factor to $SO(2l)$

is an irreducible representation of $\mathfrak{so}(2l)$ parameterized by its highest weight

$$\lambda = \mathbf{1}_l/2 + (m_1, m_2, \dots, m_l) \quad (5)$$

or

$$\bar{\lambda} = \bar{\mathbf{1}}_l/2 + (m_1, m_2, \dots, -m_l) \quad (6)$$

where $m_1 \geq \dots \geq m_l \geq 0, m_i \in \mathbb{Z}$.

There are two possibilities. First, an irreducible representation of $Pin(2l)$ factors to that of $O(2l)$, then we can use the parametrization of irreducible representations of $O(2l)$ to parameterize these representations of $Pin(2l)$.

Secondly, an irreducible representation of $Pin(2l)$ is induced from an irreducible representation of $Spin(2l)$ with highest weight of (5) or (6). When restricted to $Spin(2l)$, it will decompose into a sum of the two irreducible representations of highest weights (5) and (6). We will use $\lambda = \mathbf{1}_l/2 + (m_1, m_2, \dots, m_l), m_i \geq 0$ to denote this irreducible representation of $Pin(2l)$. Denote by

$$\Sigma(Pin) = \{\mathbf{1}_l/2 + (m_1, m_2, \dots, m_l) \mid (m_1, m_2, \dots, m_l) \in \mathcal{P}^l\}.$$

Case $\mathbf{n} = 2\mathbf{l} + 1$. An irreducible representation of $Spin(2l+1)$ which does not factor to $SO(2l+1)$ is an irreducible representation of $\mathfrak{so}(2l+1)$ parameterized by its highest weight $\lambda \in \Sigma(Pin)$.

2.2.4. $Sp(2l)$. The Lie group $Sp(2l)$ is the subgroup of $GL(2l)$ which preserves the following skew-symmetric bilinear form

$$\begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}$$

Its Lie algebra $\mathfrak{sp}(2l)$ consists of $2l \times 2l$ matrices of the following form:

$$\begin{bmatrix} a & b \\ c & -{}^t a \end{bmatrix} \quad (7)$$

where a, b, c are $l \times l$ matrices, b, c are symmetric. Let $\mathfrak{b}(\mathfrak{sp}(2l))$ be the Borel subalgebra consisting of matrices of the form (7) with $c = 0$ and a upper triangular. The Cartan subalgebra $\mathfrak{h}(\mathfrak{sp}(2l))$ consists of diagonal matrices $\text{diag}(t_1, \dots, t_l, -t_1, \dots, -t_l)$. An irreducible representation of $Sp(2l)$ can be parameterized by its highest weight $\lambda \in \Sigma(C) := \mathcal{P}^l$.

2.3. Additional notations. The notations introduced in this preliminary section are close to but do not always coincide with [W1]. For example, our b_∞ is \tilde{b}_∞ there.

We shall denote by $L(x_\infty; \Lambda)$, for $x = b, c, d$, the irreducible x_∞ -module of highest weight Λ . The level can be read off from Λ .

Given a classical Lie group G of type X , we shall denote by $V_\lambda(G)$ the irreducible G -module parameterized by $\lambda \in \Sigma(X)$. Let $W(X)$ be the Weyl group of type X .

We will denote the roots of the Lie algebra of G by standard notations $\varepsilon_i \pm \varepsilon_j, \varepsilon_i, 2\varepsilon_i$ etc, and by $(\ , \)$ the bilinear form such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Let ρ denote half the sum of positive roots.

3. CORRELATION FUNCTIONS ON d_∞ -MODULES OF LEVEL l

3.1. The Fock space \mathcal{F}^l . Let $\underline{\mathbb{Z}}$ denote $\frac{1}{2} + \mathbb{Z}$ or \mathbb{Z} , and set

$$\epsilon = \begin{cases} 0, & \text{if } \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z} \\ \frac{1}{2} & \text{if } \underline{\mathbb{Z}} = \mathbb{Z}. \end{cases}$$

Consider a pair of fermionic fields

$$\psi^+(z) = \sum_{n \in \underline{\mathbb{Z}}} \psi_n^+ z^{-n - \frac{1}{2} + \epsilon}, \quad \psi^-(z) = \sum_{n \in \underline{\mathbb{Z}}} \psi_n^- z^{-n - \frac{1}{2} + \epsilon},$$

with the following anti-commutation relations

$$[\psi_m^+, \psi_n^-]_+ = \delta_{m+n, 0}, \quad [\psi_m^\pm, \psi_n^\pm]_+ = 0.$$

Denote by \mathcal{F} the Fock space of the fermionic fields $\psi^\pm(z)$ generated by a vacuum vector $|0\rangle$ which satisfies

$$\begin{aligned} \psi_n^- |0\rangle = \psi_n^+ |0\rangle = 0 & \quad \text{for } n \in \frac{1}{2} + \mathbb{Z}_+, \quad \text{if } \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}; \\ \psi_{n+1}^- |0\rangle = \psi_n^+ |0\rangle = 0 & \quad \text{for } n \in \mathbb{Z}_+, \quad \text{if } \underline{\mathbb{Z}} = \mathbb{Z}. \end{aligned}$$

We have the standard charge decomposition (cf. [MJD])

$$\mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}^{(k)}.$$

Each $\mathcal{F}^{(k)}$ becomes an irreducible module over a certain Heisenberg Lie algebra. The shift operator $S : \mathcal{F}^{(k)} \rightarrow \mathcal{F}^{(k+1)}$ matches the highest weight vectors and commutes with the creation operators in the Heisenberg algebra.

Now we take l pairs of fermionic fields, $\psi^{\pm, p}(z)$ ($p = 1, \dots, l$) and denote the corresponding Fock space by \mathcal{F}^l . Introduce the following

generating series

$$E(z, w) \equiv \sum_{i, j \in \mathbb{Z}} E_{ij} z^{i-1+2\epsilon} w^{-j} = \sum_{p=1}^l : \psi^{+,p}(z) \psi^{-,p}(w) :, \quad (8)$$

where the normal ordering $::$ means that the operators annihilating $|0\rangle$ are moved to the right with a sign. It is well known that the operators E_{ij} ($i, j \in \mathbb{Z}$) generate a representation in \mathcal{F}^l of the Lie algebra $\widehat{\mathfrak{gl}}$ with level l .

Let

$$e_{pq}^- = \sum_{r \in \underline{\mathbb{Z}}} : \psi_r^{-p} \psi_{-r}^{-q} :, \quad e_{pq}^+ = \sum_{r \in \underline{\mathbb{Z}}} : \psi_r^{+p} \psi_{-r}^{+q} :, \quad p \neq q, \quad (9)$$

and let

$$e_{pq} = \sum_{r \in \underline{\mathbb{Z}}} : \psi_r^{+p} \psi_{-r}^{-q} : + \delta_{pq} \epsilon. \quad (10)$$

The operators $e_{pq}^+, e_{pq}, e_{pq}^-$ ($p, q = 1, \dots, l$) generate Lie algebra $\mathfrak{so}(2l)$ (cf. [FF, W1]).

3.2. The $(O(2l), d_\infty)$ -Howe duality. Now let $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ for the remainder of Section 3.

The representation of the Lie algebra d_∞ on \mathcal{F}^l is given by (cf. [DJKM2])

$$\begin{aligned} & \sum_{i, j \in \mathbb{Z}} (E_{i,j} - E_{1-j, 1-i}) z^{i-1} w^{-j} \\ &= \sum_{p=1}^l : \psi^{+,p}(z) \psi^{-,p}(w) : - : \psi^{+,p}(w) \psi^{-,p}(z) : \end{aligned} \quad (11)$$

The action of $\mathfrak{so}(2l)$ can be integrated to the action of the Lie group $SO(2l)$ on \mathcal{F}^l . In particular, the operators e_{pq} ($p, q = 1, \dots, l$) form a Lie subalgebra $\mathfrak{gl}(l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l))$ with the one generated by e_{pq}^+ ($p \neq q$), e_{pq} ($p \leq q$), $p, q = 1, \dots, l$. Note that $\tau \in O(2l)$ defined in (3) commutes with the action of d_∞ on \mathcal{F}^l . The following lemma summarizes Lemmas 3.2, 3.3 in [W1].

Lemma 3.1. *The action of the Lie group $O(2l)$ commutes with the action of d_∞ on \mathcal{F}^l .*

We define a map $\Lambda : \Sigma(D) \longrightarrow d_{\infty_0}^*$ by sending $\lambda = (m_1, \dots, m_l)$, where $m_l > 0$, to

$$\Lambda(\lambda) = (l-i)\Lambda_0^d + (l-i)\Lambda_1^d + \sum_{k=1}^i \Lambda_{m_k}^d,$$

sending $(m_1, \dots, m_j, 0, \dots, 0)$, where $j < l$, to

$$\Lambda(\lambda) = (2l - i - j)\Lambda_0^d + (j - i)\Lambda_1^d + \sum_{k=1}^i \Lambda_{m_k}^d,$$

and sending $(m_1, \dots, m_j, 0, \dots, 0) \otimes \det$, where $j < l$, to

$$\Lambda(\lambda) = (j - i)\Lambda_0^d + (2l - i - j)\Lambda_1^d + \sum_{k=1}^i \Lambda_{m_k}^d,$$

if $m_1 \geq \dots m_i > m_{i+1} = \dots = m_j = 1 > m_{j+1} = \dots = m_l = 0$.

Proposition 3.1. [W1, Theorem 3.2] *We have the following decomposition of $(O(2l), d_\infty)$ -modules:*

$$\mathcal{F}^l \cong \bigoplus_{\lambda \in \Sigma(D)} V_\lambda(O(2l)) \otimes L(d_\infty; \Lambda(\lambda)). \quad (12)$$

3.3. The main results of [BO, CW1]. Let t be an indeterminant. We define the following operators in d_∞ (cf. [W1]):

$$\begin{aligned} :D(t): &= \sum_{k \in \mathbb{N}} (t^{k-\frac{1}{2}} - t^{\frac{1}{2}-k})(E_{k,k} - E_{1-k,1-k}), \\ D(t) &= :D(t): + \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C. \end{aligned}$$

When acting on \mathcal{F}^l , these operators can be written in terms of the operators ψ_n^\pm by (11) as

$$\begin{aligned} :D(t): &= \sum_{p=1}^l \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k (: \psi_{-k}^{+,p} \psi_k^{-,p} : + : \psi_{-k}^{-,p} \psi_k^{+,p} :), \\ D(t) &= \sum_{p=1}^l \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k (\psi_{-k}^{+,p} \psi_k^{-,p} + \psi_{-k}^{-,p} \psi_k^{+,p}). \end{aligned}$$

Recall that Bloch and Okounkov [BO] introduced the following operators in $\widehat{\mathfrak{gl}}$

$$:A(t): = \sum_{k \in \mathbb{Z}} t^{k-\frac{1}{2}} E_{k,k}, \quad A(t) = :A(t): + \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C.$$

We easily verify that

$$D(t) = A(t) - A(t^{-1}), \quad :D(t): = :A(t): - :A(t^{-1}):. \quad (13)$$

Given $\lambda = (m_1, \dots, m_l) \in \Sigma(A)$, we denote by $\Lambda(\lambda)$ the $\widehat{\mathfrak{gl}}$ -highest weight $\Lambda_{m_1}^a + \dots + \Lambda_{m_l}^a$. The energy operator L_0 on the $\widehat{\mathfrak{gl}}$ -module $L(\widehat{\mathfrak{gl}}; \Lambda(\lambda))$ with highest weight vector $v_{\Lambda(\lambda)}$ is characterized by

$$\begin{aligned} L_0 \cdot v_{\Lambda(\lambda)} &= \frac{1}{2} \|\lambda\|^2 \cdot v_{\Lambda(\lambda)}, \\ [L_0, E_{ij}] &= (i - j) E_{ij}, \end{aligned} \quad (14)$$

where

$$\|\lambda\|^2 := \lambda_1^2 + \lambda_2^2 + \dots + \lambda_l^2,$$

On \mathcal{F}^l , we can realize L_0 as

$$L_0 = \sum_{p=1}^l \sum_{k \in \mathbb{Z} + \frac{1}{2}} k : \psi_{-k}^{+,p} \psi_k^{-,p} :.$$

The n -point $\widehat{\mathfrak{gl}}$ -correlation function of level l associated to λ is defined in [BO] for $l = 1$ and in [CW1] for general l as

$$\mathfrak{A}_\lambda^l(q; \mathbf{t}) \equiv \mathfrak{A}_\lambda^l(q; t_1, \dots, t_n) := \text{tr}_{L(\widehat{\mathfrak{gl}}, \Lambda(\lambda))} (q^{L_0} \mathbf{A}(t_1) \mathbf{A}(t_2) \cdots \mathbf{A}(t_n)).$$

Here and below we denote $\mathbf{t} = (t_1, \dots, t_n)$.

Let $(a; q)_\infty := \prod_{r=0}^\infty (1 - aq^r)$. Define the theta function

$$\Theta(t) := (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(q; q)_\infty^{-2} (qt; q)_\infty (qt^{-1}; q)_\infty \quad (15)$$

$$\Theta^{(k)}(t) := \left(t \frac{d}{dt} \right)^k \Theta(t), \quad \text{for } k \in \mathbb{Z}_+. \quad (16)$$

Denote by $F_{bo}(q; \mathbf{t})$ or $F_{bo}(q; t_1, \dots, t_n)$ the following expression

$$\frac{1}{(q; q)_\infty} \cdot \sum_{\sigma \in S_n} \frac{\det \left(\frac{\Theta^{(j-i+1)}(t_{\sigma(1)} \cdots t_{\sigma(n-j)})}{(j-i+1)!} \right)_{i,j=1}^n}{\Theta(t_{\sigma(1)}) \Theta(t_{\sigma(1)} t_{\sigma(2)}) \cdots \Theta(t_{\sigma(1)} t_{\sigma(2)} \cdots t_{\sigma(n)})}. \quad (17)$$

It is understood here that $1/(-k)! = 0$ for $k > 0$, and for $n = 1$, we have $F_{bo}(q; t) = (q; q)_\infty^{-1} \Theta(t)^{-1}$. The following summarizes the main results of Bloch-Okounkov [BO] for $l = 1$ and Cheng-Wang [CW1] for general $l \geq 1$.

Theorem 3.1. *Associated to $\lambda = (\lambda_1, \dots, \lambda_l)$, where $\lambda_1 \geq \dots \geq \lambda_l$ and $\lambda_i \in \mathbb{Z}$, the n -point $\widehat{\mathfrak{gl}}$ -function of level l is given by*

$$\mathfrak{A}_\lambda^l(q; \mathbf{t}) = q^{\frac{\|\lambda\|^2}{2}} (t_1 t_2 \cdots t_n)^{|\lambda|} \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) \cdot F_{bo}(q; \mathbf{t})^l$$

where $|\lambda| := \lambda_1 + \dots + \lambda_l$.

In the simplest case, i.e. $l = n = 1$, we have

$$\mathfrak{A}_\lambda^1(q; t) = q^{\frac{\lambda^2}{2}} t^\lambda \cdot F_{bo}(q; t) = \frac{q^{\frac{\lambda^2}{2}} t^\lambda}{(q; q)_\infty \Theta(t)}.$$

3.4. The 1-point d_∞ -functions of level l .

Definition 3.1. *The n -point d_∞ -correlation function of level l associated to $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{P}^l$ is*

$$\begin{aligned} \mathfrak{D}_\lambda^l(q, \mathbf{t}) &\equiv \mathfrak{D}_\lambda^l(q, t_1, \dots, t_n) \\ &:= \begin{cases} \operatorname{tr}_{L(d_\infty; \Lambda(\lambda))} q^{L_0} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n), & \text{if } \lambda_l \neq 0, \\ \operatorname{tr}_{L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))} q^{L_0} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n), & \text{if } \lambda_l = 0. \end{cases} \end{aligned}$$

(The operator L_0 is defined in the same way as for a $\widehat{\mathfrak{gl}}$ -module.)

Remark 3.1. A justification of this definition when $\lambda_l = 0$ is as follows. The two weights $\Lambda(\lambda)$ and $\Lambda(\lambda \otimes \det)$ are interchanged by a d_∞ -Dynkin diagram automorphism. Thus, the direct sum $L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))$ can be regarded as an irreducible module of the orthogonal group associated to d_∞ .

In this subsection, we will restrict to $n = 1$ for notational simplicity.

Theorem 3.2. *The 1-point d_∞ -function of level l is given by*

$$\mathfrak{D}_\lambda^l(q, t) = \frac{1}{(q; q)_\infty^l \Theta(t)^l} \cdot \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l (t^{k_a} + t^{-k_a})$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

We first prepare a few lemmas for the proof of this theorem. By a character of a module V of $SO(2l)$ or $O(2l)$, we mean $\operatorname{tr}_V(z_1^{e_{11}} \cdots z_l^{e_{ll}})$. Let $|a_{ij}|$ denote the determinant of a matrix (a_{ij}) .

Lemma 3.2. *Denote by $ch_\lambda^o(z_1, \dots, z_l)$ the character of the irreducible $O(2l)$ -module $V_\lambda(O(2l))$. If $\lambda_l = 0$, then*

$$ch_\lambda^o(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)} \right|}{\left| z_j^{l - i} + z_j^{-(l - i)} \right|}.$$

If $\lambda_l \neq 0$, then

$$ch_\lambda^o(z_1, \dots, z_l) = \frac{2 \left| z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)} \right|}{\left| z_j^{l - i} + z_j^{-(l - i)} \right|}.$$

Proof. The character of an irreducible $SO(2l)$ -module $V_\lambda(SO(2l))$, denoted by ch_λ^{so} , is well known to be as follows (cf. [FH, pp. 410]):

$$\text{ch}_\lambda^{so}(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right| + \left| z_j^{\lambda_i+l-i} - z_j^{-(\lambda_i+l-i)} \right|}{\left| z_j^{l-i} + z_j^{-(l-i)} \right|}. \quad (18)$$

Recall that an irreducible $O(2l)$ -module can be a sum of two irreducible $SO(2l)$ -modules or remain to be irreducible as a $SO(2l)$ -module, depending on whether λ_l is nonzero or not. If $\lambda_l = 0$, the second determinant in the numerator of (18) vanishes and hence the character formula for ch_λ^o of $V_\lambda(O(2l))$ follows from ch_λ^{so} . If $\lambda_l \neq 0$, then $V_\lambda(O(2l)) = V_\lambda(SO(2l)) \oplus V_{\bar{\lambda}}(SO(2l))$. Note that the second determinant terms in the numerators of ch_λ^{so} and $\text{ch}_{\bar{\lambda}}^{so}$ (cf. (18)) are opposite to each other. Now the formula for ch_λ^o follows. \square

Lemma 3.3. *We have*

$$\begin{aligned} \text{tr}_{\mathcal{F}^{(k)}} q^{L_0} \mathbf{D}(t) &= \frac{1}{2} (t^k + t^{-k}) q^{\frac{k^2}{2}} \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} \mathbf{D}(t), \quad k \in \mathbb{Z}, \\ \text{tr}_{\mathcal{F}} z^{\epsilon_{11}} q^{L_0} \mathbf{D}(t) &= \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} \mathbf{D}(t) \cdot \sum_{k \in \mathbb{Z}} \frac{(t^k + t^{-k})}{2} z^k q^{\frac{k^2}{2}}. \end{aligned}$$

Proof. These two identities are clearly equivalent, and it suffices to prove the first one. We refer to [MJD] or [Ok, Appendix A] for properties of the shift operator \mathbf{S} on \mathcal{F} . Then,

$$\begin{aligned} \text{tr}_{\mathcal{F}^{(k)}} q^{L_0} \mathbf{D}(t) &= \text{tr}_{\mathcal{F}^{(0)}} \mathbf{S}^{-k} q^{L_0} (\mathbf{A}(t) - \mathbf{A}(t^{-1})) \mathbf{S}^k \\ &= \text{tr}_{\mathcal{F}^{(0)}} q^{L_0 + \frac{k^2}{2}} (t^k \mathbf{A}(t) - t^{-k} \mathbf{A}(t^{-1})) \\ &= q^{\frac{k^2}{2}} \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} (t^k \mathbf{A}(t) - t^{-k} \mathbf{A}(t^{-1})) \end{aligned}$$

and

$$\begin{aligned} \text{tr}_{\mathcal{F}^{(-k)}} q^{L_0} \mathbf{D}(t) &= \text{tr}_{\mathcal{F}^{(0)}} \mathbf{S}^k q^{L_0} (\mathbf{A}(t) - \mathbf{A}(t^{-1})) \mathbf{S}^{-k} \\ &= \text{tr}_{\mathcal{F}^{(0)}} q^{L_0 + \frac{k^2}{2}} (t^{-k} \mathbf{A}(t) - t^k \mathbf{A}(t^{-1})) \\ &= q^{\frac{k^2}{2}} \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} (t^{-k} \mathbf{A}(t) - t^k \mathbf{A}(t^{-1})). \end{aligned}$$

By (13), $t^k \mathbf{A}(t) - t^{-k} \mathbf{A}(t^{-1}) + t^{-k} \mathbf{A}(t) - t^k \mathbf{A}(t^{-1}) = (t^k + t^{-k}) \mathbf{D}(t)$. It follows that

$$\text{tr}_{\mathcal{F}^{(-k)}} q^{L_0} \mathbf{D}(t) + \text{tr}_{\mathcal{F}^{(k)}} q^{L_0} \mathbf{D}(t) = (t^k + t^{-k}) q^{\frac{k^2}{2}} \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} \mathbf{D}(t).$$

Since $\mathcal{F}^{(-k)}$ and $\mathcal{F}^{(k)}$ are isomorphic as d_∞ -modules and $\mathbf{D}(t)$ lies in d_∞ , we have $\text{tr}_{\mathcal{F}^{(-k)}} q^{L_0} \mathbf{D}(t) = \text{tr}_{\mathcal{F}^{(k)}} q^{L_0} \mathbf{D}(t)$, and the result follows. \square

Lemma 3.4. *We have $\text{tr}_{\mathcal{F}^{(0)}} q^{L_0} \mathbf{D}(t) = 2F_{bo}(q, t) = \frac{2}{(q; q)_\infty \Theta(t)}$.*

Proof. As a special case of Theorem 3.1, we note that

$$\mathfrak{A}_{(0)}^1(q; t) = \operatorname{tr}_{\mathcal{F}(0)} q^{L_0} \mathbf{A}(t) = F_{bo}(q; t) = \frac{1}{(q, q)_\infty \Theta(t)}.$$

In particular, $\operatorname{tr}_{\mathcal{F}(0)} q^{L_0} \mathbf{A}(t^{-1}) = -\operatorname{tr}_{\mathcal{F}(0)} q^{L_0} \mathbf{A}(t)$. Thus, it follows by (13) that $\operatorname{tr}_{\mathcal{F}(0)} q^{L_0} \mathbf{D}(t) = 2F_{bo}(q; t)$. \square

Set

$$c_\lambda = \begin{cases} 1, & \text{if } \lambda_l = 0 \\ 2, & \text{if } \lambda_l \neq 0. \end{cases}$$

We have the following lemma similar to [CW1, Lemma 6].

Lemma 3.5. *We have the following identity:*

$$\prod_{i=1}^l \operatorname{tr}_{\mathcal{F}} z_i^{e_{ii}} q^{L_0} \mathbf{D}(t) = \sum_{\lambda} c_{\lambda} \cdot \frac{\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right|}{\left| z_j^{l-i} + z_j^{-(l-i)} \right|} \cdot \mathfrak{D}_{\lambda}^l(q, t).$$

Proof. It follows by computing the trace of $z_1^{e_{11}} \cdots z_l^{e_{ll}} q^{L_0} \mathbf{D}(t)$ of both sides of the $(O(2l), d_\infty)$ -duality (12). Note that the factor 2 in $\operatorname{ch}_{\lambda}^o$ in Lemma 3.2 for $\lambda_l \neq 0$ gives rise to the factor c_{λ} . \square

Let \mathbf{z}^{μ} denote $z_1^{\mu_1} \cdots z_l^{\mu_l}$ for $\mu = (\mu_1, \dots, \mu_l)$.

Lemma 3.6. *Among all the monomials \mathbf{z}^{μ} in the expansion of the determinant $\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right|$, there is exactly one “dominant” monomial with $\mu_1 \geq \dots \geq \mu_l \geq 0$, that is, $\mathbf{z}^{\lambda+\rho} = \prod_{i=1}^l z_i^{\lambda_i+l-i}$. Its coefficient is equal to $2/c_{\lambda}$, or more concretely, it is 1 if $\lambda_l \neq 0$ and 2 if $\lambda_l = 0$.*

Proof. The first part is clear by inspection. Note that the coefficient 2 when $\lambda_l = 0$ comes from the last row of the determinant. \square

Proof of Theorem 3.2. It follows by Lemmas 3.3, 3.4, and 3.5 that

$$\begin{aligned} & F_{bo}(q, t)^l \cdot \prod_{a=1}^l \left(\sum_{k_a \in \mathbb{Z}} (t^{k_a} + t^{-k_a}) z_a^{k_a} q^{\frac{k_a^2}{2}} \right) \cdot \frac{1}{2} \left| z_j^{l-i} + z_j^{-(l-i)} \right| \\ &= \sum_{\lambda} c_{\lambda}/2 \cdot \left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right| \cdot \mathfrak{D}_{\lambda}^l(q, t). \end{aligned} \quad (19)$$

Recall ([FH, (24.38)]) that the Weyl denominator of type D_l is

$$\frac{1}{2} \left| z_j^{l-i} + z_j^{-(l-i)} \right| = \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)}. \quad (20)$$

The theorem follows when we apply Lemma 3.6 to compare the coefficients of the monomial $\prod_{i=1}^l z_i^{\lambda_i+l-i}$ on both sides of (19) with (20) plugged in. \square

3.5. The n -point d_∞ -functions of level l . We now compute the n -point d_∞ -function of level l . The 1-point calculation in the previous subsection carries over for general n after suitable modification. Let

$$\mathbf{F}(z, q; t_1, \dots, t_n) = \text{tr}_{\mathcal{F}} z^{e_{11}} q^{L_0} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n) \quad (21)$$

The following lemma is the n -point generalization of Lemma 3.3.

Lemma 3.7. *We have*

$$\mathbf{F}(z, q; t_1, \dots, t_n) = \sum_{k \in \mathbb{Z}} z^k q^{\frac{k^2}{2}} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}})^k \cdot F_{bo}(q; \mathbf{t}^{\vec{\epsilon}})$$

where we denote $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, $[\vec{\epsilon}] = \epsilon_1 \epsilon_2 \cdots \epsilon_n$, $\Pi \mathbf{t}^{\vec{\epsilon}} = t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}$, and $F_{bo}(q; \mathbf{t}^{\vec{\epsilon}}) = F_{bo}(q; t_1^{\epsilon_1}, \dots, t_n^{\epsilon_n})$.

Proof. We calculate using (13) that

$$\begin{aligned} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n) &= \prod_{j=1}^n (\mathbf{A}(t_j) - \mathbf{A}(t_j^{-1})) \\ &= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} \epsilon_1 \epsilon_2 \cdots \epsilon_n \mathbf{A}(t_1^{\epsilon_1}) \mathbf{A}(t_2^{\epsilon_2}) \cdots \mathbf{A}(t_n^{\epsilon_n}) \end{aligned}$$

It is known (cf. [Ok, CW1]) by the same type of argument as in Lemma 3.3 that

$$\text{tr}_{\mathcal{F}} z^{e_{11}} q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n) = \sum_{k \in \mathbb{Z}} z^k q^{\frac{k^2}{2}} (t_1 \cdots t_n)^k F_{bo}(q, t_1, \dots, t_n).$$

Now the lemma follows. \square

Lemma 3.8. *We have the following identity:*

$$\prod_{i=1}^l \mathbf{F}(z_i, q; t_1, \dots, t_n) = \sum_{\lambda} c_{\lambda} \cdot \frac{\left| z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)} \right|}{\left| z_j^{l - i} + z_j^{-(l - i)} \right|} \cdot \mathfrak{D}_{\lambda}^l(q, t_1, \dots, t_n).$$

Proof. The lemma is a straightforward n -point version of Lemma 3.5, which is proved in the same way as before. \square

Theorem 3.3. *The n -point d_∞ -correlation function of level l associated to $\lambda \in \mathcal{P}^l$ is given by*

$$\begin{aligned} \mathfrak{D}_{\lambda}^l(q; t_1, \dots, t_n) &= \\ &= \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right) \end{aligned}$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

Proof. The proof is the same as the proof of Theorem 3.2 by using now Lemmas 3.7 and 3.8. \square

3.6. A refined 1-point function of level 1. Denote by $\mathcal{F}_{\pm}^{(0)}$ the (± 1) -eigenspace of $\tau \in O(2)$ acting on $\mathcal{F}^{(0)}$ where we recall that

$$\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As an $(O(2), d_{\infty})$ -module, we have

$$\mathcal{F} = \oplus_{m \in \mathbb{N}} (\mathcal{F}^{(m)} \oplus \mathcal{F}^{(-m)}) \oplus \mathcal{F}_+^{(0)} \oplus \mathcal{F}_-^{(0)}.$$

As d_{∞} -modules, all $\mathcal{F}^{(m)}, \mathcal{F}^{(-m)}, \mathcal{F}_+^{(0)}, \mathcal{F}_-^{(0)}$ are irreducible, and moreover, $\mathcal{F}^{(m)} \cong \mathcal{F}^{(-m)}$. The main result of this subsection is the following.

Theorem 3.4. *The $\text{tr}_{\mathcal{F}_{\pm}^{(0)}} q^{L_0} \mathcal{D}(t)$, which we refer to as a refined 1-point d_{∞} -correlation function, is equal to*

$$\frac{1}{(q; q)_{\infty} \Theta(t)} \pm (q; q^2)_{\infty} \left(\sum_{r=0}^{\infty} \left(\frac{q^{r+1} t^{-\frac{1}{2}}}{1 - q^{2(r+1)} t^{-1}} - \frac{q^{r+1} t^{\frac{1}{2}}}{1 - q^{2(r+1)} t} \right) + \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \right)$$

which is equivalent to

$$\frac{1}{(q; q)_{\infty} \Theta(t)} \pm (q; q^2)_{\infty} t \frac{d}{dt} \ln \left(\frac{(-t^{-\frac{1}{2}}; q)_{\infty} (-qt^{\frac{1}{2}}; q)_{\infty}}{(t^{-\frac{1}{2}}; q)_{\infty} (qt^{\frac{1}{2}}; q)_{\infty}} \right).$$

We denote

$$\begin{aligned} G(t) &= \text{tr}_{\mathcal{F}_+^{(0)}} q^{L_0} \mathcal{D}(t) - \text{tr}_{\mathcal{F}_-^{(0)}} q^{L_0} \mathcal{D}(t) \\ :G(t): &= \text{tr}_{\mathcal{F}_+^{(0)}} q^{L_0} : \mathcal{D}(t) : - \text{tr}_{\mathcal{F}_-^{(0)}} q^{L_0} : \mathcal{D}(t) :. \end{aligned}$$

To compute $\text{tr}_{\mathcal{F}_{\pm}^{(0)}} q^{L_0} \mathcal{D}(t)$ it suffices to compute their difference $G(t)$, since $\mathfrak{D}_{(0)}^1(t) = \text{tr}_{\mathcal{F}_+^{(0)}} q^{L_0} \mathcal{D}(t) + \text{tr}_{\mathcal{F}_-^{(0)}} q^{L_0} \mathcal{D}(t)$ has been calculated.

Recall that the *rank* $\text{rk}(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the cardinality of the set $\{i \mid \lambda_i \geq i\}$.

Lemma 3.9. *We have*

$$:G(t): = 2 \sum_{\lambda = \lambda^t} (-1)^{\text{rk}(\lambda)} q^{|\lambda|} \sum_i (t^{\lambda_i - i + 1/2} - t^{-(\lambda_i - i + 1/2)}).$$

Proof. Note that τ sends ψ_n^+ to ψ_n^- for each n and vice-versa. Thus, $\mathcal{F}_+^{(0)}$ has a basis given by

$$\begin{aligned} & \{\psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{n_1}^+ \cdots \psi_{n_r}^+ |0\rangle + \psi_{n_1}^- \cdots \psi_{n_r}^- \psi_{m_1}^+ \cdots \psi_{m_r}^+ |0\rangle \\ & \quad | m_1 < \cdots < m_r < 0, n_1 < \cdots < n_r < 0, m_i \neq n_i \text{ for some } i \} \\ \cup & \quad \{\psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{m_1}^+ \cdots \psi_{m_r}^+ |0\rangle \mid m_1 < \cdots < m_r < 0, r \text{ even}\} \end{aligned}$$

and $\mathcal{F}_-^{(0)}$ has a basis given by

$$\begin{aligned} & \{\psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{n_1}^+ \cdots \psi_{n_r}^+ |0\rangle - \psi_{n_1}^- \cdots \psi_{n_r}^- \psi_{m_1}^+ \cdots \psi_{m_r}^+ |0\rangle \\ & \quad | m_1 < \cdots < m_r < 0, n_1 < \cdots < n_r < 0, m_i \neq n_i \text{ for some } i \} \\ \cup & \quad \{\psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{m_1}^+ \cdots \psi_{m_r}^+ |0\rangle \mid m_1 < \cdots < m_r < 0, r \text{ odd}\} \end{aligned}$$

$\mathcal{F}_+^{(0)}$ (resp. $\mathcal{F}_-^{(0)}$) has highest weight vector $|0\rangle$ (resp. $\psi_{-\frac{1}{2}}^- \psi_{-\frac{1}{2}}^+ |0\rangle$).

The action of $:D(t):$ on $\mathcal{F}^{(0)}$ can be described explicitly:

$$\begin{aligned} & :D(t): \psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{n_1}^+ \cdots \psi_{n_r}^+ |0\rangle \\ & = \sum_i \left(t^{-n_i} - t^{n_i} + t^{-m_i} - t^{m_i} \right) \psi_{m_1}^- \cdots \psi_{m_r}^- \psi_{n_1}^+ \cdots \psi_{n_r}^+ |0\rangle. \end{aligned}$$

It is well known that $\mathcal{F}^{(0)}$ can be identified with an irreducible module of Heisenberg algebra and its basis is parameterized by partitions. Given an element $\psi_{-p_1}^- \cdots \psi_{-p_r}^- \psi_{-q_1}^+ \cdots \psi_{-q_r}^+ |0\rangle \in \mathcal{F}^{(0)}$, where $p_1 > \cdots > p_r > 0, q_1 > \cdots > q_r > 0$, the indices $(p_1, \dots, p_r \mid q_1, \dots, q_r)$ are exactly the Frobenius coordinates of a partition λ (which by our convention here uses half-integers). It is well known that (cf. e.g. [BO, Lemma 5.1])

$$\sum_{i=1}^l \left(t^{\lambda_i - 1 + \frac{1}{2}} - t^{-i + \frac{1}{2}} \right) = \sum_{k=1}^r \left(t^{p_k} - t^{-(-q_k)} \right).$$

We then compute that

$$\begin{aligned} \text{tr}_{\mathcal{F}_+^{(0)}} q^{L_0} :D(t): &= X + 2 \sum_{\substack{\lambda = \lambda^t \\ \text{rk}(\lambda) \text{ even}}} q^{|\lambda|} \sum_i \left(t^{\lambda_i - i + 1/2} - t^{-(\lambda_i - i + 1/2)} \right) \\ \text{tr}_{\mathcal{F}_-^{(0)}} q^{L_0} :D(t): &= X + 2 \sum_{\substack{\lambda = \lambda^t \\ \text{rk}(\lambda) \text{ odd}}} q^{|\lambda|} \sum_i \left(t^{\lambda_i - i + 1/2} - t^{-(\lambda_i - i + 1/2)} \right), \end{aligned}$$

for the same X whose explicit form is irrelevant here. Now $:G(t):$ is given by the difference of the above two formulas. \square

Proposition 3.2. *We have*

$$\begin{aligned} :G(t): &= 2(q; q^2)_\infty \cdot \sum_{n=1}^{\infty} \frac{q^{2n-1} \left(t^{-n+\frac{1}{2}} - t^{n-\frac{1}{2}} \right)}{1 - q^{2n-1}} \\ &= 2(q; q^2)_\infty \cdot \sum_{r=0}^{\infty} \left(\frac{q^{r+1} t^{-\frac{1}{2}}}{1 - q^{2(r+1)} t^{-1}} - \frac{q^{r+1} t^{\frac{1}{2}}}{1 - q^{2(r+1)} t} \right). \end{aligned}$$

and

$$G(t) = :G(t): + (q; q^2)_\infty \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.$$

Proof. There is a canonical bijection between the set of symmetric partitions (i.e. λ such that $\lambda^t = \lambda$) and the set \mathcal{OSP} of odd strict partitions. The bijection is achieved by setting the parts of a new partition μ to be the hook lengths of the diagonal nodes from the original symmetric partition λ (i.e. $\mu_i = 2\lambda_i - 2i + 1$). Here is an example:

$$(5, 4, 3, 2, 1) \mapsto (9, 5, 1).$$

Under such a bijection which sends a symmetric partition λ to $\mu \in \mathcal{OSP}$, we have $rk(\lambda) = \ell(\mu)$, $|\lambda| = |\mu|$, and $\lambda_i - i + \frac{1}{2} = \mu_i/2$. By Lemma 3.9, we have a reformulation

$$:G(t): = 2 \sum_{\mu \in \mathcal{OSP}} (-1)^{\ell(\mu)} q^{|\mu|} \sum_{k \geq 1} \left(t^{\mu_k/2} - t^{-\mu_k/2} \right). \quad (22)$$

By Theorem 8 of [W2] (replacing q therein by q^2), we have the following identity:

$$\sum_{\mu \in \mathcal{OSP}} \left(z^{\ell(\mu)} q^{|\mu|} \sum_{k \geq 1} t^{\mu_k/2} \right) = (-qz; q^2)_\infty \left(1 + \sum_{n=1}^{\infty} \frac{q^{2n-1} t^{n-\frac{1}{2}} z}{1 + q^{2n-1} z} \right).$$

Using this identity twice (with the specialization $z = -1$), we obtain from (22) the first formula for $:G(t):$. The second formula for $:G(t):$ follows from the following identity:

$$\sum_{n=1}^{\infty} \frac{z q^{2n-1} t^{n-\frac{1}{2}}}{1 + q^{2n-1} z} = \sum_{r=0}^{\infty} \frac{(-1)^r z^{r+1} q^{r+1} t^{\frac{1}{2}}}{1 - q^{2(r+1)} t}.$$

(This identity follows quickly by expanding the left side as a power series, interchanging summations, and then summing up.)

Along the same line as the proof of Lemma 3.9 and (22), we can show that

$$\mathrm{tr}_{\mathcal{F}_+^{(0)}} q^{L_0} - \mathrm{tr}_{\mathcal{F}_-^{(0)}} q^{L_0} = \sum_{\mu \in \mathcal{OSP}} (-1)^{\ell(\mu)} q^{|\mu|} = (q; q^2)_\infty$$

where the last equation is the specialization at $z = -1$ of the identity

$$\sum_{\mu \in \mathcal{OS}\mathcal{P}} z^{\ell(\mu)} q^{|\mu|} = (-qz; q^2)_\infty.$$

Now the formula for $G(t)$ follows from this consideration and (13). \square

Remark 3.2. The function $G(t)$ is essentially a “super version” of the 1-point d_∞ -correlation function of level $\frac{1}{2}$ (cf. [W2, Theorem 9]).

Proof of Theorem 3.4. We write

$$\mathrm{tr}_{\mathcal{F}_\pm^{(0)}} q^{L_0} \mathbf{D}(t) = \frac{1}{2} (\mathfrak{D}_{(0)}^1(t) \pm G(t)).$$

It is known by Theorem 3.2 that

$$\mathfrak{D}_{(0)}^1(t) = \frac{2}{(q; q)_\infty \Theta(t)}.$$

Now the first formula of the theorem follows from Proposition 3.2.

To see the equivalence of the second formula, we compute that

$$t \frac{d}{dt} \ln \left(\frac{(-t^{-\frac{1}{2}}; q)_\infty}{(t^{-\frac{1}{2}}; q)_\infty} \right) = \frac{1}{2} \sum_{r=0}^{\infty} \left(\frac{q^r t^{-\frac{1}{2}}}{1 + q^r t^{-\frac{1}{2}}} + \frac{q^r t^{-\frac{1}{2}}}{1 - q^r t^{-\frac{1}{2}}} \right).$$

Combining common denominators, one obtains the term $\frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}$ and the first group of summands of the first formula. The remaining terms in the \ln expression of the second formula yield the second group of summands of the first formula. \square

3.7. The q -dimension of a d_∞ -module of level l . Given an d_∞ -module M , we refer to $\dim_q M := \mathrm{tr}_M q^{L_0}$ as the q -dimension of M .

Proposition 3.3. *The q -dimension $\dim_q L(d_\infty; \Lambda(\lambda))$ for $\lambda_l \neq 0$, or $\dim_q [L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))]$ for $\lambda_l = 0$, is given by the following (equivalent) formulas:*

$$\begin{aligned} & \frac{1}{(q; q)_\infty^l} \cdot \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \\ &= \frac{1}{(q; q)_\infty^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j}). \end{aligned}$$

Proof. We substitute Lemma 3.3 with the simple identity

$$\mathrm{tr}_{\mathcal{F}}(z^{e_{11}} q^{L_0}) = \dim_q \mathcal{F}^{(0)} \sum_{k \in \mathbb{Z}} z^k q^{\frac{k^2}{2}},$$

and substitute Lemma 3.4 with the identity $\dim_q \mathcal{F}^{(0)} = (q; q)_\infty^{-1}$. In this way, the same strategy of establishing the 1-point function in Theorem 3.2 applies and it readily leads to the first q -dimension formula in the proposition. The second formula follows from Lemma 3.10 below and the explicit root system of D_l . \square

Lemma 3.10. *Let \mathfrak{g} be a semisimple Lie algebra with Weyl group W . Set $\|x\|^2 = (x, x)$, and let λ be a weight. Then,*

$$\sum_{\sigma \in W} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} = q^{\frac{\|\lambda\|^2}{2}} \prod_{\alpha \in \Delta^+} (1 - q^{(\lambda + \rho, \alpha)}).$$

Proof. The Weyl denominator formula reads

$$\sum_{\sigma \in W} (-1)^{\ell(\sigma)} q^{\rho - \sigma(\rho)} = \prod_{\alpha \in \Delta^+} (1 - q^\alpha).$$

The lemma now follows by applying the bilinear pairing with $\lambda + \rho$ to both sides of this formula of Weyl and noting that

$$\frac{1}{2} \|\lambda + \rho - \sigma(\rho)\|^2 = \frac{1}{2} \|\lambda\|^2 + (\lambda + \rho, \rho - \sigma(\rho)).$$

\square

4. CORRELATION FUNCTIONS ON d_∞ -MODULES OF LEVEL $l + \frac{1}{2}$

4.1. The Fock space $\mathcal{F}^{l+\frac{1}{2}}$. Recall $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ or \mathbb{Z} . Consider the neutral fermion

$$\varphi(z) = \sum_{n \in \underline{\mathbb{Z}}} \varphi_n z^{-n - \frac{1}{2} + \epsilon}$$

which satisfies the commutation relation

$$[\varphi_m, \varphi_n]_+ = \delta_{m, -n}.$$

We denote by $\mathcal{F}^{l+\frac{1}{2}}$ the Fock space of one neutral fermion $\varphi(z)$ and l pairs of complex fermions $\psi^{\pm p}(z)$, $1 \leq p \leq l$, generated by a vacuum vector $|0\rangle$ which satisfies

$$\begin{aligned} \varphi_n |0\rangle &= \psi_n^{+,p} |0\rangle = \psi_n^{-,p} |0\rangle = 0 & \text{for } n \in \frac{1}{2} + \mathbb{Z}_+, & \text{if } \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}; \\ \varphi_n |0\rangle &= \psi_n^{+,p} |0\rangle = \psi_{n+1}^{-,p} |0\rangle = 0 & \text{for } n \in \mathbb{Z}_+, & \text{if } \underline{\mathbb{Z}} = \mathbb{Z}. \end{aligned}$$

Let

$$e_p^\pm = \sum_{r \in \underline{\mathbb{Z}}} : \psi_r^{\pm p} \varphi_{-r} :, \quad 1 \leq p \leq l.$$

It is known (cf. [FF, W1]) that the above operators e_p^+, e_p^- together with $e_{pq}^+, e_{pq}^-, e_{pq}^-$ ($p, q = 1, \dots, l$) defined in (9, 10) generate Lie algebra $\mathfrak{so}(2l + 1)$.

Lemma 4.1. *Given a pair of complex fermions $\psi^\pm(z)$, we let*

$$\varphi_n := (\psi_n^+ + \psi_n^-)/\sqrt{2}, \quad \varphi'_n := i(\psi_n^+ - \psi_n^-)/\sqrt{2}.$$

Then, φ_n and φ'_n satisfy the anti-commutation relations:

$$\begin{aligned} [\varphi_n, \varphi_m]_+ &= \delta_{n,-m}, & [\varphi'_n, \varphi'_m]_+ &= \delta_{n,-m}, \\ [\varphi_n, \varphi'_m]_+ &= 0, & \text{for } m, n \in \underline{\mathbb{Z}}. \end{aligned}$$

Hence, there is an isomorphism of Fock spaces

$$\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}} \cong \begin{cases} \mathcal{F} & \text{if } \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}, \\ \mathcal{F} \oplus \mathcal{F} & \text{if } \underline{\mathbb{Z}} = \mathbb{Z}. \end{cases}$$

Proof. The commutation relations are verified by a direct computation. The multiplicity 2 in the Fock space isomorphism for $\underline{\mathbb{Z}} = \mathbb{Z}$ is due to the fact that the central elements φ_0, φ'_0 thus defined satisfy $\varphi_0|0\rangle = i\varphi'_0|0\rangle$. \square

4.2. The $(O(2l+1), d_\infty)$ -Howe duality. Now let $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ in the remainder of Section 4. The Lie algebra d_∞ acts on \mathcal{F}^l by (cf. [DJKM2])

$$\begin{aligned} & \sum_{i,j \in \mathbb{Z}} (E_{i,j} - E_{1-j,1-i}) z^{i-1} w^{-j} \\ &= \sum_{p=1}^l : \psi^{+,p}(z) \psi^{-,p}(w) : - : \psi^{+,p}(w) \psi^{-,p}(z) : + : \varphi(z) \varphi(w) :. \end{aligned} \tag{23}$$

The action of Lie algebra $\mathfrak{so}(2l+1)$ on $\mathcal{F}^{l+\frac{1}{2}}$ can be integrated to an action of the Lie group $SO(2l+1)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l+1))$ with the one generated by e_{pq}^+ ($p \neq q$), e_{pq} ($p \leq q$), e_p^+ , where $p, q = 1, \dots, l$. The element $\omega := \text{diag}(1, \dots, 1, -1) \in O(2l+1)$ acts on $\mathcal{F}^{l+\frac{1}{2}}$ by sending φ_n to $-\varphi_n$ for each n .

Lemma 4.2. [W1, Lemmas 4.2] *The action of the Lie group $O(2l+1)$ commutes with the action of d_∞ on $\mathcal{F}^{l+\frac{1}{2}}$.*

Define a map Λ from $\Sigma(B)$ to $d_{\infty 0}^*$ by sending $\lambda = (m_1, m_2, \dots, m_l)$ to

$$\Lambda(\lambda) = (2l+1-i-j)\Lambda_0^d + (j-i)\Lambda_1^d + \sum_{k=1}^i \Lambda_{m_k}^d$$

and sending $\lambda = (m_1, m_2, \dots, m_l) \otimes \det$ to

$$\Lambda(\lambda) = (j-i)\Lambda_0^d + (2l+1-i-j)\Lambda_1^d + \sum_{k=1}^i \Lambda_{m_k}^d$$

where $m_1 \geq \dots \geq m_i > m_{i+1} = \dots = m_j = 1 > m_{j+1} = \dots = m_l = 0$.

Proposition 4.1. [W1, Theorem 4.1] *We have the $(O(2l+1), d_\infty)$ -module decomposition:*

$$\mathcal{F}^{l+\frac{1}{2}} \cong \bigoplus_{\lambda \in \Sigma(B)} V_\lambda(O(2l+1)) \otimes L(d_\infty, \Lambda(\lambda)).$$

By (23), we can write $D(t)$ acting on $\mathcal{F}^{l+\frac{1}{2}}$ as

$$D(t) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k \left(\sum_{i=1}^l (\psi_{-k}^{+,i} \psi_k^{-,i} + \psi_{-k}^{-,i} \psi_k^{+,i}) + \varphi_{-k} \varphi_k \right).$$

4.3. The n -point d_∞ -function of level $\frac{1}{2}$.

Definition 4.1. *The n -point d_∞ -correlation function of level $l + \frac{1}{2}$ associated to $\lambda \in \mathcal{P}^l$, denoted by $\mathfrak{D}_\lambda^{l+\frac{1}{2}}(q, \mathbf{t})$ or $\mathfrak{D}_\lambda^{l+\frac{1}{2}}(q, t_1, \dots, t_n)$, is*

$$\mathrm{tr}_{L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))} q^{L_0} D(t_1) \cdots D(t_n).$$

(As a justification of this definition, Remark 3.1 also applies here.)

When $l = 0$, $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t_1, \dots, t_n) = \mathrm{tr}_{\mathcal{F}^{\frac{1}{2}}} q^{L_0} D(t_1) \cdots D(t_n)$, by Proposition 4.1. The aim of this subsection is to determine this (unique) n -point d_∞ -function of level $\frac{1}{2}$, which will be used in the general level $l + \frac{1}{2}$ case in the following subsection.

Lemma 4.3. *Under the isomorphism $\mathcal{F} \cong \mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}$ in Lemma 4.1, we have $D(t) = D_1(t) + D_2(t)$, where $D_1(t) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} t^k \varphi_{-k} \varphi_k$ and $D_2(t) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} t^k \varphi'_{-k} \varphi'_k$.*

Proof. A simple calculation reveals that

$$\psi_{-k}^+ \psi_k^- + \psi_{-k}^- \psi_k^+ = \varphi_{-k} \varphi_k + \varphi'_{-k} \varphi'_k.$$

Now the lemma follows from the definition of $D(t)$. \square

Given a subset $I = (i_1, \dots, i_s) \subseteq \{1, \dots, n\}$, we denote by I^c the complementary set to I , and $\mathbf{t}_I = (t_{i_1}, \dots, t_{i_s})$. By convention, we let

$$\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}_\emptyset) = \mathrm{tr}_{\mathcal{F}^{\frac{1}{2}}} q^{L_0} = (-q^{\frac{1}{2}}; q)_\infty \quad (24)$$

and recall from (21) that

$$\mathbf{F}(z, q; t_1, \dots, t_n) = \mathrm{tr}_{\mathcal{F}} z^{e_{11}} q^{L_0} D(t_1) \cdots D(t_n).$$

Proposition 4.2. *We have*

$$\mathbf{F}(1, q; t_1, \dots, t_n) = \sum_{I \subseteq \{1, \dots, n\}} \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}_I) \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}_{I^c}). \quad (25)$$

Equivalently, we have

$$\begin{aligned} \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}) = & \frac{1}{2}(-q^{\frac{1}{2}}; q)_{\infty}^{-1} \left(\sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}})^k F_{bo}(q; \mathbf{t}^{\vec{\epsilon}}) \right. \\ & \left. - \sum_{\emptyset \subsetneq I \subsetneq \{1, \dots, n\}} \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}_I) \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, \mathbf{t}_{I^c}) \right). \end{aligned}$$

Proof. By Lemmas 4.1 and 4.3, we have

$$\begin{aligned} & \text{tr}_{\mathcal{F}} q^{L_0} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n) \\ = & \text{tr}_{\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}} q^{L_0} (\mathbf{D}_1(t_1) + \mathbf{D}_2(t_1)) \cdots (\mathbf{D}_1(t_n) + \mathbf{D}_2(t_n)) \\ = & \sum_{\vec{i} \in \{1, 2\}^n} \text{tr}_{\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}} q^{L_0} \mathbf{D}_{i_1}(t_1) \mathbf{D}_{i_2}(t_2) \cdots \mathbf{D}_{i_n}(t_n). \end{aligned}$$

This is equivalent to the first formula in the theorem.

On the right-hand side of (25), there are exactly two terms equal to $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t_1, \dots, t_n)$, which come from $I = \emptyset$ and $\{1, \dots, n\}$. Now the second formula follows from (24) and Lemma 3.7 which gives a formula for $\mathbf{F}(1, q; t_1, \dots, t_n)$. \square

Proposition 4.2 allows for the determination, which is recursive on n , of all n -point correlation functions $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t_1, \dots, t_n)$. Note that the 1-point function $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t)$ has been computed in [W2] (denoted by $S(t)$ therein) using partition identities.

Proposition 4.3. [W2, Theorem 9] *The 1-point function $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t)$ is given by*

$$(-q^{\frac{1}{2}}; q)_{\infty} \left(\frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} + \sum_{r=0}^{\infty} \left[\frac{(-1)^r (q^{r+1}t)^{\frac{1}{2}}}{1 - q^{r+1}t} - \frac{(-1)^r (q^{r+1}t^{-1})^{\frac{1}{2}}}{1 - q^{r+1}t^{-1}} \right] \right).$$

An alternative solution to the 1-point function follows from Proposition 4.2 for $n = 1$:

$$\begin{aligned} \mathfrak{D}_{(0)}^{\frac{1}{2}}(q, t) &= (-q^{\frac{1}{2}}; q)_{\infty}^{-1} \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}} t^k F_{bo}(q, t) \\ &= \frac{(-q^{\frac{1}{2}}t; q)_{\infty} (-q^{\frac{1}{2}}t^{-1}; q)_{\infty}}{(-q^{\frac{1}{2}}; q)_{\infty} \Theta(t)} \\ &= \frac{1}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})} \cdot \frac{(-q^{\frac{1}{2}}t; q)_{\infty} (-q^{\frac{1}{2}}t^{-1}; q)_{\infty} (q; q)_{\infty}^2}{(qt; q)_{\infty} (qt^{-1}; q)_{\infty} (-q^{\frac{1}{2}}; q)_{\infty}} \quad (26) \end{aligned}$$

where we have used $F_{bo}(q, t^{-1}) = -F_{bo}(q, t)$ and the Jacobi triple product identity. Comparing this formula with Proposition 4.3 gives us the following.

Corollary 4.1. *The following q -identity holds:*

$$\begin{aligned} & \frac{(-q^{\frac{1}{2}}t; q)_{\infty}(-q^{\frac{1}{2}}t^{-1}; q)_{\infty}(q; q)_{\infty}^2}{(qt; q)_{\infty}(qt^{-1}; q)_{\infty}(-q^{\frac{1}{2}}; q)_{\infty}^2} \\ &= 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \sum_{r=0}^{\infty} \left[\frac{(-1)^r (q^{r+1}t)^{\frac{1}{2}}}{1 - q^{r+1}t} - \frac{(-1)^r (q^{r+1}t^{-1})^{\frac{1}{2}}}{1 - q^{r+1}t^{-1}} \right]. \end{aligned}$$

4.4. The n -point d_{∞} -functions of level $l + \frac{1}{2}$. For $\lambda \in \mathcal{P}^l$, the character of the irreducible $O(2l+1)$ -module associated to λ and $\lambda \otimes \det$ is the same, and is given as follows (cf. [FH, p. 408])

$$\text{ch}_{\lambda}^b(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})} \right|}{\left| z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})} \right|}. \quad (27)$$

The following lemma is straightforward.

Lemma 4.4. *Among all the monomials $z_1^{\mu_1} \dots z_l^{\mu_l}$ in the expansion of the determinant $\left| z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})} \right|$, there is exactly one dominant monomial with $\mu_1 \geq \dots \geq \mu_l \geq 0$, that is, $\mathbf{z}^{\lambda+\rho} = \prod_{i=1}^l z_i^{\lambda_i + l - i + \frac{1}{2}}$. Its coefficient is equal to 1.*

Recall the definition (21) of $\mathbf{F}(z, q; t_1, \dots, t_n)$.

Lemma 4.5. *We have the following q -series identity:*

$$\begin{aligned} & \text{tr}_{\mathcal{F}^{\frac{1}{2}}} q^{L_0} \mathbf{D}(t_1) \cdots \mathbf{D}(t_n) \cdot \prod_{i=1}^l \mathbf{F}(z_i, q; t_1, \dots, t_n) \\ &= \sum_{\lambda \in \mathcal{P}^l} \text{ch}_{\lambda}^b(z_1, \dots, z_l) \mathfrak{D}_{\lambda}^{l+\frac{1}{2}}(q; t_1, \dots, t_n). \end{aligned}$$

Proof. This follows from the application of $\text{tr}_{\mathcal{F}^{l+\frac{1}{2}}} z_1^{e_{11}} \cdots z_l^{e_{ll}} q^{L_0} \mathbf{D}(t)$ to both sides of the Howe duality in Proposition 4.1. Note that $\mathcal{F}^{l+\frac{1}{2}} \cong \mathcal{F}^l \otimes \mathcal{F}^{\frac{1}{2}}$. On the left-hand side, $z_i^{e_{ii}}$ only acts on the i^{th} tensor factor of \mathcal{F}^l and not on $\mathcal{F}^{\frac{1}{2}}$. For the right-hand side, $z_1^{e_{11}} \cdots z_l^{e_{ll}}$ only acts on the first tensor factor and $q^{L_0} \mathbf{D}(t)$ only acts on the second tensor factor. \square

Recall that $\mathfrak{D}_{(0)}^{\frac{1}{2}}(q; \mathbf{t})$ has been computed recursively in the previous subsection.

Theorem 4.1. *The n -point d_∞ -correlation function of level $l + \frac{1}{2}$, $\mathfrak{D}_\lambda^{l+\frac{1}{2}}(q, t_1, \dots, t_n)$, is equal to*

$$\begin{aligned} & \mathfrak{D}_{(0)}^{\frac{1}{2}}(q; \mathbf{t}) \times \\ & \times \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right) \end{aligned}$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

Proof. The Weyl denominator of type B_l reads that

$$\left| z_j^{l-i+\frac{1}{2}} + z_j^{-(l-i+\frac{1}{2})} \right| = \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)}. \quad (28)$$

It follows by (27), (28), Lemmas 3.7 and 4.5 that

$$\begin{aligned} & \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)} \cdot \mathfrak{D}_{(0)}^{\frac{1}{2}}(q; \mathbf{t}) \times \\ & \times \prod_{a=1}^l \left(\sum_{k_a \in \mathbb{Z}} z_a^{k_a} q^{\frac{k_a^2}{2}} \sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right) \\ & = \sum_{\lambda \in \mathcal{P}^l} \left| z_j^{\lambda_i + l - i + \frac{1}{2}} + z_j^{-(\lambda_i + l - i + \frac{1}{2})} \right| \cdot \mathfrak{D}_\lambda^{l+\frac{1}{2}}(q, \mathbf{t}). \end{aligned}$$

Now the theorem follows by Lemma 4.4 and by comparing the coefficients of $\mathbf{z}^{\lambda+\rho}$ on both sides of the above identity. \square

4.5. The q -dimension of a d_∞ -module of level $l + \frac{1}{2}$. In the same manner as in Section 3.7, we can derive the following q -dimension formula from the $(O(2l+1), d_\infty)$ -Howe duality in Proposition 4.1. The second formula below is obtained from the first one by using Lemma 3.10 and the explicit root system of type B_l . Recall that $L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))$ can be regarded as an irreducible module of the orthogonal group corresponding to d_∞ (cf. Remark 3.1).

Proposition 4.4. *We have*

$$\begin{aligned}
& \dim_q[L(d_\infty; \Lambda(\lambda)) \oplus L(d_\infty; \Lambda(\lambda \otimes \det))] \\
&= \frac{(-q^{-\frac{1}{2}}; q)_\infty}{(q; q)_\infty^l} \cdot \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \\
&= \frac{(-q^{-\frac{1}{2}}; q)_\infty}{(q; q)_\infty^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i \leq l} (1 - q^{\lambda_i + l - i + 1/2}) \times \\
&\quad \times \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j + 1}).
\end{aligned}$$

5. CORRELATION FUNCTIONS ON c_∞ -MODULES OF LEVEL l

5.1. The $(Sp(2l), c_\infty)$ -Howe duality. We again take $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ for the Fock space \mathcal{F}^l of fermions $\psi^{\pm, p}(z)$, $1 \leq p \leq l$, in this section. The representation of c_∞ on \mathcal{F}^l is given by ([DJKM2])

$$\begin{aligned}
& \sum_{i, j \in \mathbb{Z}} (E_{i, j} - (-1)^{i+j} E_{1-j, 1-i}) z^{i-1} w^{-j} \\
&= \sum_{p=1}^l : \psi^{+, p}(z) \psi^{-, p}(w) : - : \psi^{+, p}(w) \psi^{-, p}(z) :. \tag{29}
\end{aligned}$$

Let

$$\tilde{e}_{pq}^- = \sum_{r \in \frac{1}{2} + \mathbb{Z}} (-1)^{r - \frac{1}{2}} : \psi_r^{-, p} \psi_{-r}^{-, q} :, \quad \tilde{e}_{pq}^+ = \sum_{r \in \frac{1}{2} + \mathbb{Z}} (-1)^{r - \frac{1}{2}} : \psi_r^{+, p} \psi_{-r}^{+, q} :,$$

and let

$$\tilde{e}_{pq} = \sum_{r \in \frac{1}{2} + \mathbb{Z}} : \psi_r^{+, p} \psi_{-r}^{-, q} :.$$

The operators $\tilde{e}_{pq}^+, \tilde{e}_{pq}, \tilde{e}_{pq}^-$ ($p, q = 1, \dots, l$) generate Lie algebra $\mathfrak{sp}(2l)$ and can be integrated to the action of the Lie group $Sp(2l)$ on \mathcal{F}^l (cf. [FF, W1]). In particular, the operators \tilde{e}_{pq} ($p, q = 1, \dots, l$) form a Lie subalgebra $\mathfrak{gl}(l)$ in the horizontal of $\mathfrak{sp}(2l)$. Identify the Borel subalgebra $\mathfrak{b}(\mathfrak{sp}(2l))$ with the one generated by \tilde{e}_{pq} ($p \leq q$), \tilde{e}_{pq}^+ , $p, q = 1, \dots, l$. It is known by [W1, Lemmas 3.6, Remark 3.7] that the action of the Lie group $Sp(2l)$ commutes with the action of c_∞ on \mathcal{F}^l .

Define the map $\Lambda : \mathcal{P}^l \longrightarrow c_{\infty 0}^*$ by sending $\lambda = (m_1, \dots, m_l)$ to $\Lambda(\lambda) = (l-j)\Lambda_0^c + \sum_{k=1}^j \Lambda_{m_k}^c$, where j denotes the last non-zero index among m_i 's.

Proposition 5.1. [W1, Theorem 3.4] *We have the following decomposition of $(Sp(2l), c_\infty)$ -modules:*

$$\mathcal{F}^l = \bigoplus_{\lambda \in \mathcal{P}^l} V_\lambda(Sp(2l)) \otimes L(c_\infty; \Lambda(\lambda)). \quad (30)$$

5.2. The n -point c_∞ -functions of level l . Introduce the following operators in c_∞ :

$$\begin{aligned} :\mathbb{C}(t): &= \sum_{k \in \mathbb{N}} (t^{k-\frac{1}{2}} - t^{\frac{1}{2}-k})(E_{k,k} - E_{1-k,1-k}), \\ \mathbb{C}(t) &= :\mathbb{C}(t): + \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C. \end{aligned}$$

When acting on \mathcal{F}^l , these operators can be written in terms of the operators ψ_n^\pm by (29) as

$$\begin{aligned} :\mathbb{C}(t): &= \sum_{p=1}^l \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k (:\psi_{-k}^{+,p} \psi_k^{-,p}: + :\psi_{-k}^{-,p} \psi_k^{+,p}:), \\ \mathbb{C}(t) &= \sum_{p=1}^l \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k (\psi_{-k}^{+,p} \psi_k^{-,p} + \psi_{-k}^{-,p} \psi_k^{+,p}). \end{aligned}$$

Definition 5.1. *The n -point c_∞ -correlation function of level l associated to $\lambda \in \mathcal{P}^l$ is*

$$\mathfrak{C}_\lambda^l(q, t_1, \dots, t_n) = \text{tr}_{L(c_\infty; \Lambda(\lambda))} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n).$$

Theorem 5.1. *The n -point c_∞ -correlation function of level l is given by*

$$\mathfrak{C}_\lambda^l(q; t_1, \dots, t_n) = \sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right)$$

where $k_a = (\lambda + \rho - \sigma(\rho), \epsilon_a)$.

Proof. Note that $\mathbb{C}(t) = \mathbf{A}(t) - \mathbf{A}(t^{-1})$. The proof follows the same strategy which works for Theorems 3.2 and 3.3 for d_∞ -correlation functions of level l . We now use instead the combinatorial consequence of the $(Sp(2l), c_\infty)$ -Howe duality (30) and the character of irreducible $Sp(2l)$ -modules (cf. [FH, 24.18])

$$\text{ch}_\lambda^{sp}(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i + 1} - z_j^{-(\lambda_i + l - i + 1)} \right|}{\left| z_j^{l - i + 1} - z_j^{-(l - i + 1)} \right|}.$$

Note that the Weyl group $W(C_l)$ replaces $W(D_l)$ in the proof and result. \square

In the case $n = 1$, the notation can be much simplified. The 1-point c_∞ -function of level l is given by

$$\mathfrak{C}_\lambda^l(q, t) = F_{bo}(q, t)^l \cdot \sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l (t^{k_a} + t^{-k_a})$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

Let us specialize further to $l = 1$. The irreducible character of $Sp(2) = SL(2)$ is simply

$$\text{ch}_m^{sp}(z) = (z^{m+1} - z^{-(m+1)}) / (z - z^{-1}).$$

Then the 1-point c_∞ -correlation function of level 1 is given by

$$\mathfrak{C}_{(m)}^1(q, t) = \frac{q^{m^2/2} (t^m + t^{-m}) - q^{(m+2)^2/2} (t^{m+2} + t^{-(m+2)})}{(q; q)_\infty \Theta(t)}.$$

In contrast to the d_∞ case at level 1 where the charge decomposition of \mathcal{F} and the theory of partitions can be used effectively, the description of irreducible c_∞ -submodules in \mathcal{F} is not explicit and the Howe duality in Proposition 5.1 is essentially used.

5.3. The q -dimension of a c_∞ -module. In the same manner as in Section 3.7, we can derive the following q -dimension formula from the $(Sp(2l), c_\infty)$ -Howe duality in Proposition 5.1. The second formula below is obtained from the first one by using Lemma 3.10 and the explicit root system of type C_l .

Proposition 5.2. *For $\lambda \in \mathcal{P}^l$, we have*

$$\begin{aligned} \dim_q L(c_\infty; \Lambda(\lambda)) &= \frac{1}{(q; q)_\infty^l} \cdot \sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \\ &= \frac{1}{(q; q)_\infty^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i \leq l} (1 - q^{2(\lambda_i + l - i + 1)}) \times \\ &\quad \times \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j + 2}). \end{aligned}$$

6. CORRELATION FUNCTIONS ON b_∞ -MODULES OF LEVEL l

6.1. The $(Pin(2l), b_\infty)$ -Howe duality. Throughout Section 6 we take $\underline{\mathbb{Z}} = \mathbb{Z}$. The action of b_∞ on the Fock space \mathcal{F}^l is given by ([DJKM2])

$$\sum_{i,j \in \mathbb{Z}} (E_{i,j} - E_{-j,-i}) z^i w^{-j} = \sum_{p=1}^l (:\psi^{+,p}(z)\psi^{-,p}(w): - :\psi^{+,p}(w)\psi^{-,p}(z):). \quad (31)$$

The Lie algebra $\mathfrak{so}(2l)$ defined in Section 3.2 can be integrated to $Spin(2l)$ and then naturally extended to $Pin(2l)$. Remark 3.5 and Lemma 3.5 of [W1] are summed up by the following lemma.

Lemma 6.1. *The action of the Lie group $Pin(2l)$ commutes with the action of b_∞ on \mathcal{F}^l .*

For $\lambda = \mathbf{1}_l/2 + (m_1, \dots, m_l)$ in $\Sigma(Pin)$, define the following map $\Lambda : \Sigma(Pin) \longrightarrow b_{\infty 0}^*$:

$$\Lambda(\lambda) = (2l - 2j)\Lambda_0^b + \sum_{k=1}^j \Lambda_{m_k}^b$$

where j is such that $m_1 \geq \dots \geq m_j > m_{j+1} = \dots = m_l = 0$.

Proposition 6.1. [W1, Theorem 3.3] *We have the following decomposition of $(Pin(2l), b_\infty)$ -modules:*

$$\mathcal{F}^l = \bigoplus_{\lambda \in \Sigma(Pin)} V_\lambda(Pin(2l)) \otimes L(b_\infty; \Lambda(\lambda)).$$

6.2. The operator $B(t)$. Introduce the following operators in b_∞ :

$$\begin{aligned} :B(t): &= \sum_{k \in \mathbb{Z}_+} (t^k - t^{-k})(E_{k,k} - E_{-k,-k}), \\ B(t) &= :B(t): + \frac{t+1}{t-1}C. \end{aligned}$$

When acting on \mathcal{F}^l , $:B(t):$ and $B(t)$ can be expressed using (31) as follows:

$$\begin{aligned} :B(t): &= \sum_{p=1}^l \sum_{k \in \mathbb{Z}} t^k (:\psi_{-k}^{+,p}\psi_k^{-,p}: + :\psi_{-k}^{-,p}\psi_k^{+,p}:), \\ B(t) &= \sum_{p=1}^l \sum_{k \in \mathbb{Z}} t^k (\psi_{-k}^{+,p}\psi_k^{-,p} + \psi_{-k}^{-,p}\psi_k^{+,p}). \end{aligned}$$

We easily verify that

$$\mathbf{B}(t) = t^{\frac{1}{2}}\mathbf{A}(t) - t^{-\frac{1}{2}}\mathbf{A}(t^{-1}), \quad :\mathbf{B}(t): = t^{\frac{1}{2}}:\mathbf{A}(t): - t^{-\frac{1}{2}}:\mathbf{A}(t^{-1}):. \quad (32)$$

The energy operator L_0 on the b_∞ -module $L(b_\infty, \Lambda(\lambda))$ with highest weight vector $v_{\Lambda(\lambda)}$ is defined by (14) and

$$[L_0, E_{i,j} - E_{-j,-i}] = (i - j)(E_{i,j} - E_{-j,-i}).$$

On \mathcal{F}^l , we can realize L_0 as

$$L_0 = \sum_{p=1}^l \sum_{k \in \mathbb{Z}} k : \psi_{-k}^{+,p} \psi_k^{-,p} : + \frac{l}{8}. \quad (33)$$

6.3. The n -point b_∞ -functions of level l .

Definition 6.1. *The n -point b_∞ -correlation function of level l associated to $\lambda = \mathbf{1}_l/2 + (m_1, \dots, m_l) \in \Sigma(\text{Pin})$ is*

$$\mathfrak{B}_\lambda^l(q, \mathbf{t}) \equiv \mathfrak{B}_\lambda^l(q, t_1, \dots, t_n) = \text{tr}_{L(b_\infty; \Lambda(\lambda))} q^{L_0} \mathbf{B}(t_1) \cdots \mathbf{B}(t_n).$$

Lemma 6.2. *Let $\lambda \in \Sigma(\text{Pin})$ and denote by $\text{ch}_\lambda^{\text{pin}}(z_1, \dots, z_l)$ the character of $V_\lambda(\text{Pin}(2l))$. Then,*

$$\text{ch}_\lambda^{\text{pin}}(z_1, \dots, z_l) = \frac{2 \left| z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)} \right|}{\left| z_j^{l - i} + z_j^{-(l - i)} \right|}.$$

Proof. Recall that $\bar{\lambda} = \bar{\mathbf{1}}_l/2 + (m_1, \dots, m_{l-1}, -m_l)$. Since $V_\lambda(\text{Pin}(2l)) \cong V_\lambda(\text{Spin}(2l)) \oplus V_{\bar{\lambda}}(\text{Spin}(2l))$, we have $\text{ch}_\lambda^{\text{pin}} = \text{ch}_\lambda^{\text{so}} + \text{ch}_{\bar{\lambda}}^{\text{so}}$, where $\text{ch}_\lambda^{\text{so}}$ for $\lambda \in \Sigma(\text{Pin})$ is also given by (18). Note that the second determinant terms in the numerators of $\text{ch}_\lambda^{\text{so}}$ and $\text{ch}_{\bar{\lambda}}^{\text{so}}$ (cf. (18)) are opposite to each other. Now the formula for $\text{ch}_\lambda^{\text{pin}}$ follows. \square

Let

$$\mathbf{F}_b(z, q; t_1, \dots, t_n) := \text{tr}_{\mathcal{F}} z^{e_{11}} q^{L_0} \mathbf{B}(t_1) \cdots \mathbf{B}(t_n). \quad (34)$$

Lemma 6.3. *We have*

$$\mathbf{F}_b(z, q; t_1, \dots, t_n) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} z^k q^{\frac{k^2}{2}} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}})^k F_{bo}(q; \mathbf{t}^{\vec{\epsilon}})$$

where the notations are as in Lemma 3.7.

Proof. The proof is similar to the one for Lemma 3.7, while we have to take into account the difference coming from the *integral* indices on the fermions $\psi^\pm(z)$. Denote the charge operator $C = \sum_{k \in \mathbb{Z}} : \psi_{-k}^+ \psi_k^- :$ which

acts as 0 on $\mathcal{F}^{(0)}$. By definition, $e_{11} = C + \frac{1}{2}$. We can check (compare [MJD] and [Ok, Appendix A]) that

$$S^{-k}e_{11}S^k = e_{11} + k, \quad S^{-k}A(t)S^k = t^k A(t),$$

and

$$S^{-k}L_0S = L_0 + kC + \frac{1}{2}k(k+1) = L_0 + ke_{11} + \frac{1}{2}k^2.$$

Then, using (34) and (32), we have

$$\begin{aligned} & \mathbf{F}_b(z, q; t_1, \dots, t_n) \\ &= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] (\Pi \mathbf{t}^{\vec{\epsilon}})^{\frac{1}{2}} \sum_{k \in \mathbb{Z}} \text{tr}_{\mathcal{F}^{(k)}} z^{e_{11}} q^{L_0} A(t_1^{\epsilon_1}) \cdots A(t_n^{\epsilon_n}) \\ &= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] (\Pi \mathbf{t}^{\vec{\epsilon}})^{\frac{1}{2}} \sum_{k \in \mathbb{Z}} \text{tr}_{\mathcal{F}^{(0)}} S^{-k} z^{e_{11}} q^{L_0} A(t_1^{\epsilon_1}) \cdots A(t_n^{\epsilon_n}) S^k \\ &= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] (\Pi \mathbf{t}^{\vec{\epsilon}})^{\frac{1}{2}} \sum_{k \in \mathbb{Z}} z^{k+\frac{1}{2}} q^{\frac{1}{2}k(k+1)} \text{tr}_{\mathcal{F}^{(0)}} q^{L_0} (\Pi \mathbf{t}^{\vec{\epsilon}})^k A(t_1^{\epsilon_1}) \cdots A(t_n^{\epsilon_n}) \\ &= \sum_{z \in \mathbb{Z}} z^{k+\frac{1}{2}} q^{\frac{1}{2}(k+\frac{1}{2})^2} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}})^{k+\frac{1}{2}} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}}). \end{aligned}$$

In the last equation we have used the “correction term” in (33). \square

Lemma 6.4. *We have the following identity:*

$$\prod_{i=1}^l \mathbf{F}_b(z_i, q; t_1, \dots, t_n) = \sum_{\lambda \in \Sigma(\text{Pin})} ch_{\lambda}^{pin}(z_1, \dots, z_l) \cdot \mathfrak{B}_{\lambda}^l(q, t_1, \dots, t_n).$$

Proof. Follows from the $(\text{Pin}(2l), b_{\infty})$ -Howe duality in Proposition 6.1. \square

Theorem 6.1. *The n -point b_{∞} -correlation function of level l associated to $\lambda \in \Sigma(\text{Pin})$ is given by*

$$\begin{aligned} & \mathfrak{B}_{\lambda}^l(q; t_1, \dots, t_n) = \\ & \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right) \end{aligned}$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

Proof. Note that Lemma 3.6 on the dominant monomial of the numerator of ch_{λ}^{pin} remains valid for $\lambda = \mathbf{1}_l/2 + (m_1, \dots, m_l)$. Now the proof of the theorem is the same as for Theorems 3.2 and 3.3, with the help of Lemmas 6.2, 6.3, and 6.4. \square

Remark 6.1. It is remarkable that the formula for \mathfrak{B}_λ^l in Theorem 6.1 coincides with the one for \mathfrak{D}_λ^l given in Theorem 3.3, except that the λ used in these two cases are different.

6.4. The q -dimension of a b_∞ -module. In the same manner as in Section 3.7, we can derive the following q -dimension formula from the $(Pin(2l), b_\infty)$ -Howe duality in Proposition 5.1.

Proposition 6.2. *For $\lambda \in \Sigma(Pin)$, we have*

$$\begin{aligned} \dim_q L(b_\infty; \Lambda(\lambda)) &= \frac{1}{(q; q)_\infty^l} \cdot \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \\ &= \frac{1}{(q; q)_\infty^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j}). \end{aligned}$$

7. CORRELATION FUNCTIONS ON b_∞ -MODULES OF LEVEL $l + \frac{1}{2}$

7.1. The $(Spin(2l + 1), b_\infty)$ -Howe duality. Let $\mathbb{Z} = \mathbb{Z}$ throughout this section. The action of Lie algebra $\mathfrak{so}(2l + 1)$ on the Fock space $\mathcal{F}^{l+\frac{1}{2}}$ defined in Section 4.1 can be integrated to an action of Lie group $Spin(2l + 1)$. The action of b_∞ on the Fock space $\mathcal{F}^{l+\frac{1}{2}}$ is given by

$$\begin{aligned} &\sum_{i,j \in \mathbb{Z}} (E_{i,j} - E_{-j,-i}) z^i w^{-j} \\ &= \sum_{p=1}^l : \psi^{+,p}(z) \psi^{-,p}(w) : - : \psi^{+,p}(w) \psi^{-,p}(z) : + : \varphi(z) \varphi(w) :. \quad (35) \end{aligned}$$

Now the operator $B(t)$ acting on $\mathcal{F}^{l+\frac{1}{2}}$ can be written as

$$B(t) = \sum_{p=1}^l \sum_{k \in \mathbb{Z}} t^k (\psi_{-k}^{+,p} \psi_k^{-,p} + \psi_{-k}^{-,p} \psi_k^{+,p}) + t^k \varphi_{-k} \varphi_k.$$

Define $\Lambda' : \Sigma(Pin) \rightarrow b_{\infty 0}^*$ by sending $\lambda = \frac{1}{2} \mathbf{1}_l + (m_1, m_2, \dots, m_l)$ to

$$\Lambda'(\lambda) = (2l + 1 - 2j) \Lambda_0^b + \sum_{k=1}^j \Lambda_{m_k}^b$$

if $m_1 \geq \dots \geq m_j > m_{j+1} = \dots = m_l = 0$.

Proposition 7.1. [W1, Theorem 4.2] *We have the $(Spin(2l + 1), b_\infty)$ -module decomposition:*

$$\mathcal{F}^{l+\frac{1}{2}} \cong 2 \bigoplus_{\lambda \in \Sigma(Pin)} V_\lambda(Spin(2l + 1)) \otimes L(b_\infty, \Lambda'(\lambda))$$

where the factor 2 denotes the multiplicity.

The energy operator L_0 on the b_∞ -module $L(b_\infty, \Lambda'(\lambda))$ with highest weight vector $v_{\Lambda'(\lambda)}$ is defined by

$$\begin{aligned} L_0 \cdot v_{\Lambda'(\lambda)} &= \left(\frac{1}{2} \|\lambda\|^2 + \frac{1}{16} \right) \cdot v_{\Lambda'(\lambda)}, \\ [L_0, E_{i,j} - E_{-j,-i}] &= (i - j)(E_{i,j} - E_{-j,-i}). \end{aligned}$$

The convention of shift by $\frac{1}{16}$ will be convenient later on, and it also fits with the standard realization in terms of neutral fermions with integral indices (i.e. Ramond sector) of L_0 of the Virasoro algebra.

7.2. The n -point b_∞ -function of level $\frac{1}{2}$.

Definition 7.1. *The n -point b_∞ -correlation function of level $l + \frac{1}{2}$ associated to $\lambda \in \Sigma(\text{Pin})$ is*

$$\mathfrak{B}_\lambda^{l+\frac{1}{2}}(q, t_1, \dots, t_n) = \text{tr}_{L(b_\infty; \Lambda'(\lambda))} q^{L_0} \mathbf{B}(t_1) \cdots \mathbf{B}(t_n).$$

On $\mathcal{F}^{l+\frac{1}{2}}$, we can realize L_0 as

$$L_0 = \sum_{p=1}^l \sum_{k \in \mathbb{Z}} k : \psi_{-k}^{+p} \psi_k^{-p} : + \sum_{k \in \mathbb{Z}} \frac{k}{2} : \varphi_{-k} \varphi_k : + \frac{2l+1}{16}. \quad (36)$$

When $l = 0$, we have by Proposition 7.1 that

$$\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t_1, \dots, t_n) = \frac{1}{2} \text{tr}_{\mathcal{F}^{\frac{1}{2}}} q^{L_0} \mathbf{B}(t_1) \cdots \mathbf{B}(t_n). \quad (37)$$

The aim of this subsection is to determine this (unique) n -point b_∞ -function of level $\frac{1}{2}$ parallel to Section 4.3, which will be used in the general level $l + \frac{1}{2}$ case in the following subsection.

The following lemma is straightforward once we recall the setup of Lemma 4.1.

Lemma 7.1. *Under the isomorphism $2\mathcal{F} \cong \mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}$ in Lemma 4.1, we have $\mathbf{B}(t) = \mathbf{B}_1(t) + \mathbf{B}_2(t)$, where $\mathbf{B}_1(t) = \sum_{k \in \mathbb{Z}} t^k \varphi_{-k} \varphi_k$ and $\mathbf{B}_2(t) = \sum_{k \in \mathbb{Z}} t^k \varphi'_{-k} \varphi'_k$.*

By convention, we let

$$\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}_\emptyset) = \frac{1}{2} \text{tr}_{\mathcal{F}^{\frac{1}{2}}} q^{L_0} = q^{\frac{1}{16}} (-q; q)_\infty. \quad (38)$$

Proposition 7.2. *Recalling \mathbf{F}_b from (34), we have*

$$\mathbf{F}_b(1, q; t_1, \dots, t_n) = 2 \sum_{I \subseteq \{1, \dots, n\}} \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}_I) \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}_{I^c}). \quad (39)$$

Equivalently, $\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t})$ is given by

$$\begin{aligned} & \frac{1}{2} q^{-\frac{1}{16}} (-q; q)_{\infty}^{-1} \left(\frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{\frac{k^2}{2}} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}})^k F_{bo}(q; \mathbf{t}^{\vec{\epsilon}}) \right. \\ & \quad \left. - \sum_{\emptyset \subsetneq I \subsetneq \{1, \dots, n\}} \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}_I) \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}_{I^c}) \right). \end{aligned}$$

Proof. By Lemmas 4.1 and 7.1, we have

$$\begin{aligned} & 2 \operatorname{tr}_{\mathcal{F}} q^{L_0} \mathbf{B}(t_1) \cdots \mathbf{B}(t_n) \\ &= \operatorname{tr}_{\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}} q^{L_0} (\mathbf{B}_1(t_1) + \mathbf{B}_2(t_1)) \cdots (\mathbf{B}_1(t_n) + \mathbf{B}_2(t_n)) \\ &= \sum_{\vec{i} \in \{1, 2\}^n} \operatorname{tr}_{\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}} q^{L_0} \mathbf{B}_{i_1}(t_1) \mathbf{B}_{i_2}(t_2) \cdots \mathbf{B}_{i_n}(t_n). \end{aligned}$$

This is equivalent to the first formula in the theorem by (37), since the definitions of L_0 on \mathcal{F} and on $\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}}$ are compatible (see (33) and (36)).

On the right-hand side of (39), there are exactly two terms from $I = \emptyset$ and $\{1, \dots, n\}$ which give rise to $\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t_1, \dots, t_n)$. Now the second formula follows from (38) and Lemma 6.3. \square

Proposition 7.2 allows for the determination, which is recursive on n , of all n -point correlation functions $\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t_1, \dots, t_n)$ of level $\frac{1}{2}$. The 1-point function $\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t)$ has been computed in [W2] (denoted by $R(t)$ therein up to a factor $q^{\frac{1}{16}}$) using partition identities.

Proposition 7.3. [W2, Theorem 4] *The 1-point function $\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t)$ is given by*

$$q^{\frac{1}{16}} (-q; q)_{\infty} \left(\frac{t+1}{2(t-1)} + \sum_{r=0}^{\infty} \left[\frac{(-1)^r q^{r+1} t}{1 - q^{r+1} t} - \frac{(-1)^r q^{r+1} t^{-1}}{1 - q^{r+1} t^{-1}} \right] \right).$$

An alternative solution to the 1-point function follows from Proposition 7.2 for $n = 1$:

$$\begin{aligned} \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, t) &= \frac{1}{4q^{\frac{1}{16}}(-q; q)_{\infty}} \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{\frac{k^2}{2}} (t^k + t^{-k}) F_{bo}(q, t) \\ &= \frac{q^{\frac{1}{8}} t^{\frac{1}{2}} (-qt; q)_{\infty} (-t^{-1}; q)_{\infty} (q; q)_{\infty}}{2q^{\frac{1}{16}}(-q; q)_{\infty} \cdot t^{\frac{1}{2}}(1 - t^{-1})\Theta(t)(q; q)_{\infty}} \\ &= \frac{q^{\frac{1}{16}}(-qt; q)_{\infty} (-t^{-1}; q)_{\infty} (q; q)_{\infty}^2}{2(qt; q)_{\infty} (t^{-1}; q)_{\infty} (-q; q)_{\infty}} \end{aligned}$$

where we have used a version of the Jacobi triple product identity

$$\sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{\frac{k^2}{2}} t^k = q^{\frac{1}{8}} t^{\frac{1}{2}} (q; q)_{\infty} (-qt; q)_{\infty} (-t^{-1}; q)_{\infty}.$$

Comparing the two formulas of 1-point function, we have the following.

Corollary 7.1. *The following q -identity holds:*

$$\begin{aligned} &\frac{(-qt; q)_{\infty} (-t^{-1}; q)_{\infty} (q; q)_{\infty}^2}{(qt; q)_{\infty} (t^{-1}; q)_{\infty} (-q; q)_{\infty}^2} \\ &= \frac{t+1}{t-1} + 2 \sum_{r=0}^{\infty} \left[\frac{(-1)^r q^{r+1} t}{1 - q^{r+1} t} - \frac{(-1)^r q^{r+1} t^{-1}}{1 - q^{r+1} t^{-1}} \right]. \end{aligned}$$

The right hand side of the identity is known (cf. [W2]) to be equal to

$$2t \frac{d}{dt} \ln \left(t^{-\frac{1}{2}} \frac{(t; q^2)_{\infty} (q^2 t^{-1}; q^2)_{\infty}}{(qt; q^2)_{\infty} (qt^{-1}; q^2)_{\infty}} \right).$$

7.3. The n -point b_{∞} -functions of level $l + \frac{1}{2}$. The character of the irreducible $Spin(2l+1)$ -module associated to $\lambda \in \Sigma(Pin)$ is also given by $ch_{\lambda}^b(z_1, \dots, z_l)$ in (27).

Lemma 7.2. *We have the following q -series identity:*

$$\begin{aligned} &\mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q, \mathbf{t}) \cdot \prod_{i=1}^l \mathbf{F}_b(z_i, q; t_1, \dots, t_n) \\ &= \sum_{\lambda \in \Sigma(Pin)} ch_{\lambda}^b(z_1, \dots, z_l) \cdot \mathfrak{B}_{\lambda}^{l+\frac{1}{2}}(q; t_1, \dots, t_n). \end{aligned}$$

Proof. Follows from the Howe duality in Proposition 7.1 and (37). Note that the cancellation of a factor 2 has occurred. \square

Theorem 7.1. *The n -point b_∞ -correlation function of level $l + \frac{1}{2}$, $\mathfrak{B}_\lambda^{l+\frac{1}{2}}(q, t_1, \dots, t_n)$, is equal to*

$$\begin{aligned} & \mathfrak{B}_{(\frac{1}{2})}^{\frac{1}{2}}(q; \mathbf{t}) \times \\ & \times \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\varepsilon}_a \in \{\pm 1\}^n} [\vec{\varepsilon}_a] (\Pi \mathbf{t}^{\vec{\varepsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\varepsilon}_a}) \right) \end{aligned}$$

where $k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a)$.

Sketch of a proof. The proof is completely parallel to the proof for Theorem 4.1, now with the help of (27), (28), Lemmas 4.4, 6.3 and 7.2. \square

7.4. The q -dimension of a b_∞ -module of level $l + \frac{1}{2}$. In the same manner as in Section 3.7, we can derive the following q -dimension formula from the Howe duality in Proposition 7.1. The second formula below is obtained from the first one by using Lemma 3.10 and the explicit root system of type B_l .

Proposition 7.4. *For $\lambda \in \Sigma(\text{Pin})$, we have*

$$\begin{aligned} & \dim_q L(b_\infty; \Lambda'(\lambda)) \\ & = q^{\frac{1}{16}} \frac{(-q; q)_\infty}{(q; q)_\infty^l} \cdot \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^2}{2}} \\ & = q^{\frac{1}{16}} \frac{(-q; q)_\infty}{(q; q)_\infty^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i \leq l} (1 - q^{\lambda_i + l - i + 1/2}) \times \\ & \quad \times \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j + 1}). \end{aligned}$$

REFERENCES

- [An] G. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its applications **2**, Addison-Wesley, 1976.
- [Blo] S. Bloch, *Zeta values and differential operators on the circle*, J. Algebra **182** (1996), 476–500.
- [BO] S. Bloch and A. Okounkov, *The characters of the infinite wedge representation*, Adv. in Math. **149** (2000), 1–60.
- [BtD] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer-Verlag.
- [CL] S.-J. Cheng and N. Lam, *Infinite-dimensional Lie superalgebras and hook Schur functions*, Commun. Math. Phys. **238** (2003), 95–118.
- [CW1] S.-J. Cheng and W. Wang, *The Bloch-Okounkov correlation functions at higher levels*, Transformation Groups **9** (2004), 133–142.
- [CW2] ———, *The correlation functions of vertex operators and Macdonald polynomials*, J. Algebraic Combin. **25** (2007), 43–56.

- [DJKM1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Operator approach to the Kadomtsev-Petviashvili equation. Transformation groups for soliton equations III*, J. Phys. Soc. Japan **50** (1981), 3806–3812.
- [DJKM2] ———, *A new hierarchy of soliton equations of KP-type. Transformation groups for soliton equations IV*, Physics **4D** (1982), 343–365.
- [FF] A. Feingold and I. Frenkel, *Classical affine algebras*, Adv. in Math. **56** (1985), 117–172.
- [FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang, $W_{1+\infty}$ and $W(gl_N)$ with central charge N , Commun. Math. Phys. **170** (1995), 337–357.
- [Fr] I. Frenkel, *Representations of affine Lie algebras, Hecke modular forms and Kortweg-de Vries type equations*, Lect. Notes. Math. **933** (1982), 71–110.
- [FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer, 1991.
- [Ho1] R. Howe, *Remarks on classical invariant theory*, Trans. AMS **313** (1989), 539–570.
- [Ho2] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Schur Lect. (Tel Aviv)(1992), 1–182, Israel Math. Conf. Proc. **8**.
- [Kac] V. Kac, *Vertex algebras for beginners*, second edition, University Lecture Series **10**, AMS, Providence, RI, 1998.
- [KR] V. Kac and A. Radul, *Representation theory of the vertex algebra $W_{1+\infty}$* , Transformation Groups, Vol. **1** (1996), 41–70.
- [KWY] V. Kac, C.H. Yan, and W. Wang, *Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$* , Adv. in Math. **139** (1998), 56–140.
- [Lep] J. Lepowsky, *Application of a “Jacobi identity” for vertex operator algebras to zeta values and differential operators*, Lett. Math. Phys. **53** (2000), 87–103.
- [LQW] W.-P. Li, Z. Qin and W. Wang, *Hilbert schemes, integrable hierarchies, and Gromov-Witten theory*, Internat. Math. Res. Notices **40** (2004), 2085–2104.
- [Mil] A. Milas, *Formal differential operators, vertex operator algebras and zeta-values, II*, J. Pure Appl. Algebra **183** (2003), 191–244.
- [MJD] T. Miwa, M. Jimbo and E. Date, *Solitons. Differential equations, symmetries and infinite dimensional algebras*, (originally published in Japanese 1993), Cambridge University Press, 2000.
- [Ok] A. Okounkov, *Infinite wedge and random partitions*, Select. Math., New Series **7** (2001), 1–25.
- [OP] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and completed cycles*, Ann. of Math. (2) **163** (2006), 517–560.
- [W1] W. Wang, *Duality in infinite dimensional Fock representations*, Commun. Contemp. Math. **1** (1999), 155–199.
- [W2] ———, *Correlation functions of strict partitions and twisted Fock spaces*, Transformation Groups **9** (2004), 89–101.
- [Zhu] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), 237–302.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE,
VA 22904

E-mail address: dgtaylor@virginia.edu

E-mail address: ww9c@virginia.edu